The Moduli Space of Curves, with Applications to Enumerative Geometry

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Abstract

Our aim is to prove Kontsevich's formula for rational plane curves, which states that the number N_d of rational plane curves of degree d passing $3d-1$ fixed general points in the complex projective plane satisfies the following recurrence relationship.

$$N_d + \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-1} \cdot d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_B = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-2} \cdot d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B$$

When accompanied with the base case $N_1 = 1$, the formula enables the computation of all numbers N_d.

The proof uses the fine moduli space of stable n-pointed rational curves and the coarse moduli space of Kontsevich stable n-pointed maps, which are used to parametrise the rational plane curves that feature in the formula. We introduce the concepts of moduli spaces, of stable curves and of stable maps in our exposition, and sketch the construction of the fine moduli space. We also define Weil divisors, and use them to analyse the boundaries of the moduli spaces, which is an important part of the proof. We largely follow the exposition of our main reference, Kock and Vainsencher's book 'Kontsevich's Formula for Rational Plane Curves' [4]. Our conclusion explains how the proof we provide is somewhat typical of modern enumerative geometry.

Contents

Prerequisites

This dissertation has been written for the reader who has been introduced to some elementary algebraic geometry, and uses notions such as quasi-projective varieties and function fields. We do not assume knowledge of schemes or divisors, unlike other texts on this subject. In relation to the undergraduate degree at Oxford University, the C3.4 Algebraic Geometry course covers the required material.

1 Introduction

One of the core properties of the complex projective plane $\mathbb{P}^2$ is that through every pair of distinct points there is a unique line. A similar statement holds for conics in $\mathbb{P}^2$: through every set of five distinct general points, there passes a unique conic. The conics may be parametrised by their coefficients $[\mathbf{a}] \in \mathbb{P}^5$,

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4yz + a_5zx + a_6xy = 0, \quad [x : y : z] \in \mathbb{P}^2$$

and this natural five-dimensional parametrisation suggests that the number of points required to determine the conic is indeed five.

We want to ask how many rational degree-d curves pass an appropriate number of points in the plane, and we will set this number to N_d. It turns out that the correct number of (general) points we need to supply is $3d - 1$, and this corresponds to a $(3d-1)$-dimensional parametrisation of rational curves using their coefficients, as for conics above. The rational degree-1 curves are simply the lines, and the rational degree-2 curves are the conics, so we have just found that $N_1 = 1$ and $N_2 = 1$.

The focus of this dissertation is the following theorem due to Kontsevich which gives a recursive formula which can be used to compute N_d for all d.

Theorem (Kontsevich, [4, Theorem 3.3.1])**.** *Let N_d be the number of rational curves of degree d passing through $3d-1$ general points in the plane $\mathbb{P}^2$. Then $N_1 = 1$ and for all d,*

$$\begin{aligned} N_d \;+ \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-1} \cdot d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_B \\ = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-2} \cdot d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B \end{aligned} \tag{1}$$

This formula yields

$$\begin{aligned} N_2 &= 1, \\ N_3 &= 12, \\ N_4 &= 620, \\ N_5 &= 87304, \\ N_6 &= 26312976, \\ &\dots \end{aligned}$$

1.1 Parametrisations

A key property of an irreducible rational curve is that it can be parametrised by the projective line $\mathbb{P}^1$. Instead of looking at the equation which defines the curve, we will use Kontsevich's approach and look at the maps which parametrise the curve.

We want our notion of the degree of a map to correspond to the degree of the curve it parametrises. This means we have to introduce a slightly different definition of the degree of a map.

Definition 1.1. [4, 2.1] The degree of the map $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ is the product $d \cdot e$ of the degree of the image curve d and the degree of the field extension e corresponding to the map.

Example 1.2. Take the map $\mu_1 : \mathbb{P}^1 \to \mathbb{P}^2, [x : y] \mapsto [x^2 : xy : y^2]$ with image curve $C_1 = \mathbb{V}(uw - v^2) \subset \mathbb{P}^2$ of degree $d = 2$. Then $C_1 - [1 : 0 : 0]$ is isomorphic to the open affine set $\mathbb{A}^1$ via $[v^2 : v : 1] \leftrightarrow v$, hence the function field is $\mathbb{C}(C_1) \cong \mathbb{C}(v)$. The degree of the corresponding field extension $\mathbb{C}(v) \to \mathbb{C}(x), v \mapsto x$ is $e = 1$, and thus the degree of the map μ_1 is $2 \cdot 1 = 2$.

Next, consider the map $\mu_2 : \mathbb{P}^1 \to \mathbb{P}^2, [x : y] \mapsto [x^2 : x^2 : y^2]$, which has image curve $C_2 = \mathbb{V}(u - v) \subset \mathbb{P}^2$ of degree $d = 1$. Now, $C_2 - [1 : 1 : 0] \cong \mathbb{A}^1, [u : u : 1] \leftrightarrow u$, and $\mathbb{C}(C_2) \cong \mathbb{C}(u)$. The field extension is now $\mathbb{C}(u) \to \mathbb{C}(x), u \mapsto x^2$ with degree $e = 2$. Therefore, the degree of μ_2 is $1 \cdot 2 = 2$ also. In some sense, it is more natural to consider the image curve as the double line $\mathbb{V}((u - v)^2)$.

As suggested by Example 1.2, a degree-d map $\mathbb{P}^1 \to \mathbb{P}^r$ is indeed determined, up to constant factor, by $r + 1$ binary forms of degree d that do not simultaneously vanish. This condition defines an open subset $W \subset \mathbb{P}^{(r+1)(d+1)-1}$, and those maps which are birational onto their image constitute an open subset of W [4, Proposition 2.1.8]. We will see W later in Section 3.4.

We want to identify the curves passing through fixed points $P_1, \ldots P_n \in \mathbb{P}^2$. To do this, we will study maps $(\mu : \mathbb{P}^1 \to \mathbb{P}^2; p_1, \ldots, p_n)$ with specified points $p_i \in \mathbb{P}^1$ in the source space, and we will count the curves with $\mu(p_i) = P_i$. Our first task is then to study collections of points in $\mathbb{P}^1$, and this is the objective of Section 2. In Section 3, we will study the maps with the accompanying points in the source space. Section 4 studies the structure that will be fundamental to our proof in Section 5 of Kontsevich's formula.

2 Stable Curves

Consider four ordered distinct points $p_1, \ldots, p_4 \in \mathbb{P}^1, p_i = [x_i : y_i]$ on the projective line. Recall that any ordered triple of distinct points in $\mathbb{P}^1$ is linearly equivalent to the standard triple $(0, 1, \infty)$ where

$$0 = [0 : 1], 1 = [1 : 1], \infty = [1 : 0]$$

It follows that any $(p_i)_{i=1}^4$ is equivalent to $(0, 1, \infty, \lambda)$ for a unique $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$, where λ is the cross-ratio of the points, given by

$$\lambda = [(x_2y_3 - x_3y_2)(x_4y_1 - x_1y_4) : (x_2y_1 - x_1y_2)(x_4y_3 - x_3y_4)]$$

In the above way, ordered quadruples of distinct points in $\mathbb{P}^1$ are naturally parametrised by their cross-ratio in $\mathbb{P}^1 - \{0, 1, \infty\}$, a quasi-projective variety. At the moment, our parametrisation consists only of a bijection between (equivalence classes of) ordered quadruples and $\mathbb{P}^1 - \{0, 1, \infty\}$, but we want to be able to use the structure of this parameter space.

2.1 Families of curves

The definitions and results in this section are taken from [4, Sections 0 and 1], and the arguments are similar to those provided by Kock and Vainsencher.

Definition 2.1. An n-pointed smooth rational curve

$$(C; p_1, \ldots, p_n)$$

is a projective smooth rational curve C with n distinct points $p_1, \ldots, p_n \in C$ called marks. An isomorphism between two n-pointed rational curves

$$\phi : (C; p_1, \ldots, p_n) \xrightarrow{\sim} (C'; p'_1, \ldots, p'_n)$$

is an isomorphism of curves $\phi : C \xrightarrow{\sim} C'$ such that $\phi(p_i) = p'_i$ for each i.

In order to find the geometric structure of the set of isomorphism classes of n-pointed rational curves, we will need to start considering families of curves as follows.

Definition 2.2. A family of n-pointed smooth rational curves is a flat and proper map $\pi : \mathfrak{X} \to B$ with n disjoint sections $\sigma_i : B \to \mathfrak{X}$ such that each geometric fibre $\mathfrak{X}_b = \pi^{-1}(b)$ is a projective smooth rational curve.

Proper and flat are technical conditions which ensure the family 'behaves well'. We shall not focus on these conditions, and they are included for precision only. The word rational shall be implicit henceforth when discussing curves. Either of the notational conventions $\mathfrak{X} \to B$ or $\mathfrak{X}/B$ may be used for a family. The sections σ_i and projection π will often be implicit, and the same symbols may refer to the sections or projections of different families.

Notice that for each $b \in B$, the fibre $(\mathfrak{X}_b; \sigma_1(b), \ldots, \sigma_n(b))$ is an n-pointed smooth rational curve, where the sections σ_i identify the marks on these curves.

Definition 2.3. An isomorphism between two families $\pi : \mathfrak{X} \to B$ and $\pi' : \mathfrak{X}' \to B$ over the same space B is an isomorphism of quasi-projective varieties $\mathfrak{X} \to \mathfrak{X}'$ such that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d, shift left, "\pi"] & \mathfrak{X}' \arrow[d, shift right, "\pi'"'] \\ B \arrow[u, shift left, "\sigma_1,\ldots,\sigma_n"] \arrow[r, equal] & B \arrow[u, shift right, "\sigma'_1,\ldots,\sigma'_n"'] \end{tikzcd}$$

It is a fact that every smooth rational curve is isomorphic to $\mathbb{P}^1$. Therefore, for $n \geq 3$, each fibre is isomorphic to the n-pointed rational curve $(\mathbb{P}^1; 0, 1, \infty, \lambda_1, \ldots, \lambda_{n-3})$ for unique points $\lambda_1, \ldots, \lambda_{n-3} \in \mathbb{P}^1$. Given any isomorphism from the fibre to $\mathbb{P}^1$ under which the marks are mapped to the points $p_1, \ldots, p_n$, and an isomorphism of $\mathbb{P}^1$ taking p_1, p_2, p_3 to the standard triple $0, 1, \infty$, the mark λ_i is the cross-ratio of p_1, p_2, p_3, p_{i+3}. The isomorphism is uniquely determined because the first three sections fix the isomorphism when they are set to $0, 1, \infty$. Although we have only shown the existence of a bijection, these isomorphisms of fibres in fact yield an isomorphism of families

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d, shift left, "\pi"] & B \times \mathbb{P}^1 \arrow[d, shift right] \\ B \arrow[u, shift left, "\sigma_1,\ldots,\sigma_n"] \arrow[r, equal] & B \arrow[u, shift right, "0,1,\infty,\lambda_1,\ldots,\lambda_{n-3}"'] \end{tikzcd}$$

Thus the family $\mathfrak{X} \to B$ is determined up to isomorphism by the maps $\lambda_1, \ldots, \lambda_{n-3} : B \to \mathbb{P}^1$. This is the idea that motivates the definition of a moduli space below. First, we need to define a pull-back in the category of families of n-pointed curves.

Definition 2.4. The pull-back from a family $\pi : \mathfrak{X} \to B$ along a morphism $\psi : B' \to B$ is the unique family $\pi' : \mathfrak{X}' \to B'$ such that π' is the usual pull-back of

$$\begin{tikzcd} & \mathfrak{X} \arrow[d, "\pi"] \\ B' \arrow[r, "\psi"] & B \end{tikzcd}$$

and the sections $\sigma_i' : B' \to \mathfrak{X}'$ are chosen so that the diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] & \mathfrak{X} \\ B' \arrow[r] \arrow[u, "\sigma_1', \ldots, \sigma_n'"] & B \arrow[u, "\sigma_1, \ldots, \sigma_n"'] \end{tikzcd}$$

commutes. Explicitly, we have

$$\begin{aligned} \mathfrak{X}' &= \{(b', x) \in B' \times \mathfrak{X} \mid \pi(x) = \psi(b')\} \\ \pi'((b', x)) &= b' \\ \sigma_i'(b') &= (b', \sigma_i(\psi(b'))) \end{aligned}$$

Definition 2.5. The universal family is the family $U \to M$ such that every other family $\mathfrak{X} \to B$ is induced, up to isomorphism, via the pull-back construction by a unique morphism $B \to M$. It is defined up to isomorphism, as with all universal property definitions. The space M is called the fine moduli space.

Moduli spaces are typically defined in a more category-theoretic way; see [3, Section 2.1.3]. Our definition is tailored to our problem, and seeing how this works will make this definition clearer.

Take any family $\mathfrak{X} \to B$ and let $\lambda_1, \ldots, \lambda_{n-3} : B \to \mathbb{P}^1$ be the sections we found before. The universal family will capture the information in the sections $\lambda_1, \ldots, \lambda_{n-3}$. Therefore, we set the fine moduli space to be the possible values of $\lambda_1, \ldots, \lambda_{n-3}$ as follows.

$$M = M_{0,n} = (\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \{\text{diagonals}\} \tag{2}$$

The diagonals are disallowed because the points must be distinct. Moreover none of the points can be $0, 1, \infty$ since these are the values of the first three sections.

Set $U = U_{0,n} = M_{0,n} \times \mathbb{P}^1$. The universal family is $U_{0,n} \to M_{0,n}$, with sections given by

$$\begin{aligned} \tau_1(\boldsymbol{\lambda}) &= (\boldsymbol{\lambda}, 0), & \tau_3(\boldsymbol{\lambda}) &= (\boldsymbol{\lambda}, \infty) \\ \tau_2(\boldsymbol{\lambda}) &= (\boldsymbol{\lambda}, 1), & \tau_i(\boldsymbol{\lambda}) &= (\boldsymbol{\lambda}, \lambda_{i-3}) \end{aligned}$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-3}) \in M_{0,n}$. We can check that the family $B \times \mathbb{P}^1 \to B$ is the family induced by the pull-back along

$$b \in B \mapsto (\lambda_1(b), \ldots, \lambda_{n-3}(b)) \in M_{0,n}$$

It is easy to see the uniqueness of this map, for no two fibres of our family $U_{0,n} \to M_{0,n}$ are isomorphic, and hence the image of $b \in B$ must be the point of $M_{0,n}$ whose fibre in $U_{0,n}$ is isomorphic to the fibre of b.

Observe that for $n = 4$, we have $M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ as suggested at the start of the section.

We have therefore seen that

Theorem 2.6. [4, Proposition 1.1.2] *For $n \geq 3$, there is a fine moduli space $M_{0,n}$ for the problem of classifying n-pointed smooth rational curves up to isomorphism.*

2.2 Stable Curves

While $M_{0,n}$ is a smooth quasi-projective variety, we can extend our definition of curve to get a fine moduli space that is a projective variety. In our proof of Kontsevich's formula, this 'compactification' of $M_{0,n}$ will be crucial. In Section 4, we will see how the improved structure may be exploited. The following definitions are taken from [4, Section 1.2]. Let $n \geq 3$ throughout.

Definition 2.7. A tree of projective lines is a connected curve such that

1. Each irreducible component is isomorphic to a projective line.
2. The points of intersection of the components are ordinary double points.
3. There are no closed circuits, meaning that if a node is removed, the curve becomes disconnected.

An irreducible component of the curve is called a twig.

Definition 2.8. A stable n-pointed curve $(C; p_1, \ldots, p_n)$ is a tree C of projective lines with n distinct smooth points $p_i \in C$ called marks, such that every twig of C has at least three special points. A special point is either a mark or a node.

Definition 2.9. A family of stable n-pointed curves is a flat and proper map $\pi : \mathfrak{X} \to B$ with n disjoint sections $\sigma_i : B \to \mathfrak{X}$, such that every geometric fibre $\mathfrak{X}_b = \pi^{-1}(b)$ is a stable n-pointed curve, whose marks are given by the evaluation of the sections.

Note that, since the marks are smooth points of the curves, the sections are disjoint from the singular points (nodes) of the fibres. Naturally, an isomorphism of stable curves is an isomorphism of curves which preserves the marks.

Example 2.10. For $n \geq 3$, any n-pointed smooth rational curve $(\mathbb{P}^1; p_1, \dots, p_n)$ is automatically a stable n-pointed curve.

Example 2.11. When discussing stable n-pointed curves, we will use diagrams like those in Figure 1. Here, each line segment corresponds to an isomorphic copy of $\mathbb{P}^1$, and the dots indicate where the marks are. Note that there may be many curves which have the structure of twigs and distribution of marks of any given diagram, so the diagrams do not necessarily uniquely identify curves.

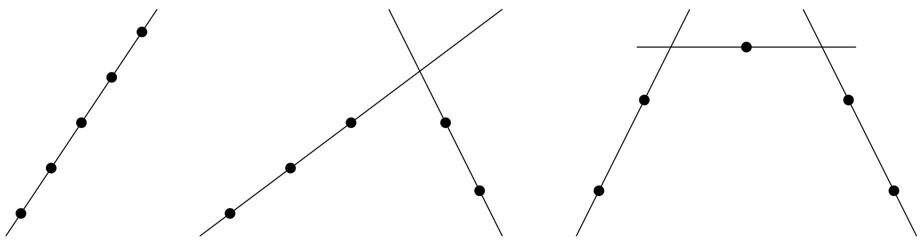

Figure 1: [4] The possible configuration of twigs and marks for 5-pointed curves.

It seems unintuitive that in a family of stable n-pointed curves, the number of twigs in the curve can be different for different curves in the family. Example 2.12 will provide a family with this property, and will be referred to later in the project.

Example 2.12. Let $B = \mathbb{C} - \{1\}$ and set

$$\mathfrak{X} = \{(b, [s:t], [x:y:z]) \in B \times \mathbb{P}^1 \times \mathbb{P}^2 \mid by = x, sz = tx\} \tag{3}$$

We will write coordinates of $\mathfrak{X}$ as $(b, \mathbf{s}, \mathbf{x})$. Let $\pi : \mathfrak{X} \to B$ be given by $\pi(b, \mathbf{s}, \mathbf{x}) = b$, and define the sections $\sigma_0, \sigma_1, \sigma_\infty, \sigma_\Delta : B \to \mathfrak{X}$ by

$$\begin{aligned} \sigma_0(b) &= (b, [0:1], [0:0:1]) \\ \sigma_1(b) &= (b, [1:1], [b:1:b]) \\ \sigma_\infty(b) &= (b, [1:0], [b:1:0]) \\ \sigma_\Delta(b) &= (b, [b:1], [b:1:1]) \end{aligned} \tag{4}$$

Notice that, in this notation, $\sigma_\alpha(b)$ has $\mathbf{s}$-coordinate α, for $\alpha = 0, 1, \infty$, and in the first two coordinates, σ_Δ is the diagonal map. Example 2.16 motivates this notation more strongly. The maps have all been given in terms of polynomials in their coordinates, so there can be no dispute that these maps are morphisms of quasi-projective varieties.

For $b \neq 0$, the fibre is

$$\mathfrak{X}_b = \pi^{-1}(b) = \left\{(b, [s:t], [sb:s:tb]) \mid [s:t] \in \mathbb{P}^1\right\} \tag{5}$$

which is isomorphic to $\mathbb{P}^1$ via a projection from the $\mathbf{s}$-coordinate. The marks are $0, 1, \infty$ and $[b:1]$, which are distinct points if $b \neq 0, 1$. This is why we removed the point 1 from B.

When $b = 0$, the fibre is

$$\begin{aligned} \mathfrak{X}_0 = \pi^{-1}(0) = &\left\{(0, [s:t], [0:1:0]) \mid [s:t] \in \mathbb{P}^1\right\} \\ &\cup \left\{(0, [0:1], [0:y:z]) \mid [y:z] \in \mathbb{P}^1\right\} \end{aligned} \tag{6}$$

Let S and X be the components in which $\mathbf{s}$ and $\mathbf{x}$ vary respectively. The fibre $\mathfrak{X}_0$ thus contains two isomorphic copies S, X of $\mathbb{P}^1$ which intersect at

$$v = (0, [0:1], [0:1:0]) \in S \cap X$$

Figure 2 shows the distribution of the marks on these two components, and thus that the fibre is a stable 4-pointed curve.

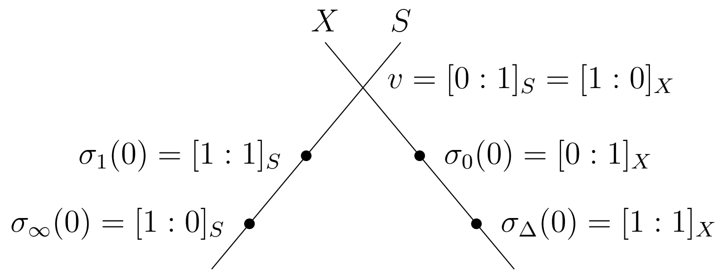

Figure 2: The fibre at $b = 0$ is a stable 4-pointed curve with two irreducible components.

The stability condition (Definition 2.8) corresponds precisely to the curve being free of non-trivial automorphisms, so the only automorphism of the curve is the identity morphism. The following argument is adapted from [4, 1.2.4].

We first prove that for any automorphism, each twig maps to itself and each node maps to itself. This is done by induction on the number of twigs, letting n vary. Every curve has a twig c with zero or one node, because there are no closed circuits. Then c is mapped to itself, since it has two or more marks. If c has a node, then since the node is the only non-singular point of c, it too must be mapped to itself. Then remove the twig, replace the node with a mark, and proceed by induction. Finally, every twig has three or more special points that are mapped to themselves, so the isomorphism restricted to each twig is the identity, for three points uniquely determine an isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$. In particular, notice that any twig with fewer than three special points would admit a non-trivial automorphism, so the stability condition is indeed required.

2.3 Removing marks

One very natural question to ask is whether we can just remove, say, the last of the marks of a stable curve to get another stable curve. Formally, take a stable $(n+1)$-pointed curve $(C; p_1, \dots, p_{n+1})$, and consider the n-pointed curve $(C; p_1, \dots, p_n)$ given by removing the last mark p_{n+1}. Checking our definition of a stable curve, we see that the new n-pointed curve is automatically a stable curve, unless the mark p_{n+1} was needed to ensure that every twig has at least three special points.

Consider the case where p_{n+1} was needed for C to be stable. Let c be the twig containing p_{n+1}, which must have precisely two other special points. Since $n \geq 3$ by assumption, one of the two points must be a node. Therefore, we have two cases to consider – either both of these points are nodes, or one is a node and the other is a mark. In both cases, we contract the twig c to a point. If one of the points was a mark, we define this mark to be the point to which c was contracted. See Figure 3 to see how this works in each case. Given the curve $(C; p_1, \dots, p_n)$, the new curve is called the *contraction* of C, and is obtained by *forgetting* p_{n+1}.

This procedure of removing a mark and contracting the resulting curve if it is unstable may be done for a family, as stated in Theorem 2.13.

Theorem 2.13 (Adapted from [4, Proposition 1.3.4]). *Let $\pi : \mathfrak{X} \to B$ be a family of stable $(n+1)$-pointed curves with sections $\sigma_1, \dots, \sigma_{n+1}$. Then there exists a family $\pi' : \mathfrak{X}' \to B$ of stable n-pointed curves with sections $\sigma'_1, \dots, \sigma'_n$ and a morphism of quasi-projective varieties $\phi : \mathfrak{X} \to \mathfrak{X}'$ such that*

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\phi"] \arrow[d, shift left, "\pi"] & \mathfrak{X}' \arrow[d, shift right, "\pi'"'] \\ B \arrow[u, shift left, "\sigma_1,\dots,\sigma_n"] \arrow[r, equal] & B \arrow[u, shift right, "\sigma'_1,\dots,\sigma'_n"'] \end{tikzcd}$$

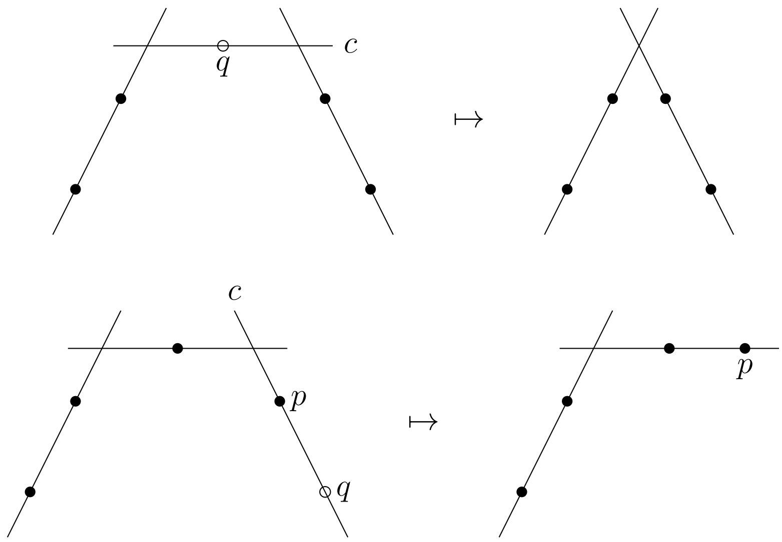

Figure 3: [4, 1.3.3] The mark q has been removed from each of these 5-pointed curves to produce an unstable 4-pointed curve. We contract each curve to produce a new stable 4-pointed curve. The two cases of contraction each involve contracting the twig c containing q to a point, which then becomes either a node or a mark.

commutes (notice the section σ_{n+1} does not appear in the diagram), and moreover that the induced morphism $\mathfrak{X}_b \to \mathfrak{X}'_b$ on the fibres restricts to an isomorphism on any stable twig of $(\mathfrak{X}_b; \sigma_1(b), \ldots, \sigma_n(b))$ and contracts any unstable twig to a point. The family $\mathfrak{X}' \to B$ is unique up to isomorphism with these properties.

Example 2.14. The family $\pi : B \times \mathbb{P}^1 \to B$ with canonical sections $\tau_\alpha(b) = (b, \alpha)$ for $\alpha = 0, 1, \infty$ is the family obtained by forgetting the fourth section σ_Δ from the family defined in Example 2.12. The associated morphism $\phi : \mathfrak{X} \to B \times \mathbb{P}^1$ is given by

$$\phi(b, \mathbf{s}, \mathbf{x}) = (b, \mathbf{s})$$

We can quickly verify the relevant properties. Commutativity of the diagram follows from

$$\pi(b, \mathbf{s}, \mathbf{x}) = (\pi \circ \phi)(b, \mathbf{s}, \mathbf{x}) = b \qquad \tau_\alpha(b) = \phi \circ \sigma_\alpha(b) = (b, \alpha)$$

For $b \neq 0$, every fibre consists of one twig which remains stable after removing the fourth mark. The morphism induced is an isomorphism because $\mathfrak{X}_b$ was seen to be isomorphic to $\mathbb{P}^1$ via the $\mathbf{s}$-coordinate projection.

The interesting case is thus $b = 0$, when $\mathfrak{X}_0 = S \cup X$. The same argument as for $b \neq 0$ will apply to S, giving an isomorphism $S \to \pi^{-1}(b) \subset B \times$

$\mathbb{P}^1$. Furthermore, the entire twig X, which becomes unstable after removing $\sigma_\Delta(0)$, is mapped to $\infty \in S$.

2.4 Adding marks

The procedure of forgetting a section has a natural inverse called *stabilisation*. Given a stable n-pointed curve $(C; p_1, \ldots, p_n)$ and an additional point $q \in C$, we consider the $(n+1)$-pointed curve $(C; p_1, \ldots, p_n, q)$. Straight from the definition of stable curve, we notice there is only one situation in which the new curve is not stable, namely when the point q coincides with another mark or node of the original curve. To produce a new stable curve, we must insert a new twig on which the new mark q will reside. See Figure 4 to see how this works in each case.

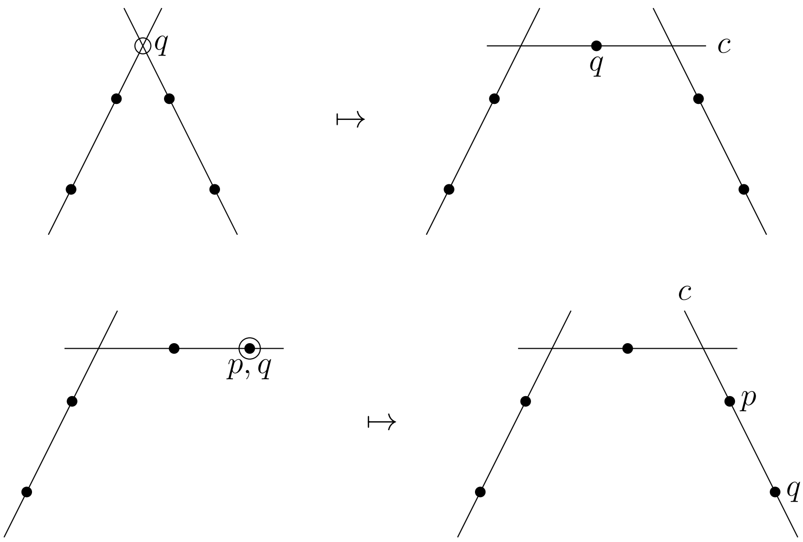

Figure 4: [4, 1.3.1] We have introduced a new mark q into each of these stable 4-pointed curves, but it coincides with a special point of the curve. We therefore stabilise each curve to produce a new stable 5-pointed curve. Each of the cases involves adding a new twig c which has precisely three special points on it. Therefore, up to isomorphism, it doesn't matter where the points lie on c, so long as they are distinct.

Theorem 2.15 describes how the procedure of stabilising a curve with an additional section is characterised for families of curves.

Theorem 2.15 (Adapted from [4, Proposition 1.3.2]). *Let $\pi : \mathfrak{X} \to B$ be a family of stable n-pointed curves with sections $\sigma_1, \ldots, \sigma_n$. Let $\delta : B \to \mathfrak{X}$ be an arbitrary additional section. Then there exists a family $\pi' : \mathfrak{X} \to B$*

of stable $(n+1)$-pointed curves with sections $\sigma'_1, \dots, \sigma'_{n+1}$ and a morphism $\phi : \mathfrak{X}' \to \mathfrak{X}$ such that

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "\phi"] \arrow[d, shift left, "\pi'"] & \mathfrak{X} \arrow[d, shift right, "\pi"'] \\ B \arrow[u, shift left, "{\sigma'_1,\dots,\sigma'_n,\sigma'_{n+1}}"] \arrow[r, equal] & B \arrow[u, shift right, "{\sigma_1,\dots,\sigma_n,\delta}"'] \end{tikzcd}$$

commutes, and its restriction to $\phi^{-1}(\mathfrak{X} - \delta(B)) \to \mathfrak{X} - \delta(B)$ is an isomorphism. The family $\pi' : \mathfrak{X}' \to B$ is unique up to isomorphism with these properties.

Example 2.16. We will now show that the family in Example 2.12 is the stabilisation of the family of 3-pointed curves over $B = \mathbb{C} - \{1\}$ given by $B \times \mathbb{P}^1 \to B$ and sections $\tau_\alpha(b) = (b, \alpha)$ for $\alpha = 0, 1, \infty$, with the new section $\delta(b) = (b, [b:1])$. The morphism $\phi : \mathfrak{X} \to B \times \mathbb{P}^1$ is given by $(b, \mathbf{s}, \mathbf{x}) \mapsto (b, \mathbf{s})$. We have the following diagram.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\phi"] \arrow[d, shift left, "\pi"] & B \times \mathbb{P}^1 \arrow[d, shift right, "\pi"'] \\ B \arrow[u, shift left, "{\sigma_0,\sigma_1,\sigma_\infty,\sigma_\Delta}"] \arrow[r, equal] & B \arrow[u, shift right, "{\tau_0,\tau_1,\tau_\infty,\delta}"'] \end{tikzcd}$$

Commutativity from the diagram is immediate from

$$\pi(b, \mathbf{s}, \mathbf{x}) = \pi \circ \phi(b, \mathbf{s}, \mathbf{x}) = b \qquad \begin{aligned} \phi \circ \sigma_\alpha(b) &= \tau_\alpha(b) = (b, \alpha) \\ \phi \circ \sigma_\Delta(b) &= \delta(b) = (b, [b:1]) \end{aligned}$$

for $\alpha = 0, 1, \infty$. Furthermore,

$$\begin{aligned} B \times \mathbb{P}^1 - \delta(B) &= \big\{(b, [s:t]) \in B \times \mathbb{P}^1 \mid [s:t] \neq [b:1]\big\} \\ \phi^{-1}(B \times \mathbb{P}^1 - \delta(B)) &= \{(b, [s:t], [x:y:z]) \in \mathfrak{X} \mid [s:t] \neq [b:1]\} \\ &= \{(b, [s:t], [sb:s:tb]) \in \mathfrak{X} \mid [s:t] \neq [b:1]\} \end{aligned}$$

and ϕ is certainly an isomorphism when restricted to this domain.

2.5 Moduli space of stable curves

We saw that n-pointed curves (Definition 2.1) are automatically stable n-pointed curves in Example 2.10. In Theorem 2.17 however, we state the result that motivates the definition of stable curves as an extension of n-pointed curves in Section 2.1.

Theorem 2.17. [4, Theorem 1.2.5] *For $n \geq 3$, there is a fine moduli space $\overline{M}_{0,n}$ for stable n-pointed curves which is a smooth projective variety and contains $M_{0,n}$ as a dense open subset.*

This means that there is a universal family for stable n-pointed curves $U \to M$, so that every family of stable n-pointed curves is induced by a pull-back from this family. We showed in Theorem 2.6 that there is a universal family $U_{0,n} \to M_{0,n}$ for n-pointed curves, which is therefore isomorphic to the pull-back from a unique morphism $M_{0,n} \to M$. This morphism naturally embeds $M_{0,n}$ in M, and Theorem 2.17 states that the image, naturally identified with $M_{0,n}$, is a dense open subset of M. It is for this reason that we denote this universal family as $\overline{U}_{0,n} \to \overline{M}_{0,n}$.

We will not go through the construction of this moduli space in detail, for this is beyond the scope of this dissertation. For a more complete sketch construction, the reader should see [4, Section 1.4]. It is a fact, however, that the universal family is given by the forgetful map $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$, and the moduli spaces may be constructed recursively. In Example 2.18, we argue that we already know $\overline{M}_{0,3}$, which will form the base case of the recursion. Example 2.19 then goes through the first step of the recursion and constructs $\overline{M}_{0,4}$. Note that these examples only explain the construction of the spaces $\overline{M}_{0,3}$ and $\overline{M}_{0,4}$. Though the same argument constructs $\overline{M}_{0,n}$ for general $n \geq 3$, we are not proving that we have found the moduli space, or any other of the properties claimed in Theorem 2.17.

Example 2.18. We will do the $n = 3$ case for this induction argument. Any stable 3-pointed curve is isomorphic to the standard curve $(\mathbb{P}^1; 0, 1, \infty)$. Therefore $\overline{M}_{0,3} = M_{0,3}$ is the family $\mathbb{P}^1 \to \{*\}$ with sections $0, 1, \infty$. Given Theorem 2.6, it is enough to convince yourself that every stable 3-pointed curve is simply a 3-pointed curve from Section 2.1.

Example 2.19. Consider the pull-back of

$$\begin{tikzcd} & \mathbb{P}^1 \arrow[d, shift right, "\pi"'] \\ \mathbb{P}^1 \arrow[r, "\pi"] & \{*\} \arrow[u, shift right, "\sigma_1,\sigma_2,\sigma_3"'] \end{tikzcd}$$

which yields a unique family over $\mathbb{P}^1$ up to isomorphism, which we will denote $\mathbb{P}^1 \times_{\{*\}} \mathbb{P}^1 \to \mathbb{P}^1$. We have a natural further section of this family given by

$$\delta : \mathbb{P}^1 \to \mathbb{P}^1 \times_{\{*\}} \mathbb{P}^1 \qquad \delta(x) = (x, x)$$

We then have the ability to construct a new family of stable 4-pointed curves with base space $\mathbb{P}^1$ by stabilising the family with this new section δ using

Theorem 2.15. This yields a family $\mathfrak{X} \to \mathbb{P}^1$ with sections $\tau_1, \dots, \tau_4$, and a morphism $\phi : \mathfrak{X} \to \mathbb{P}^1 \times_{\{*\}} \mathbb{P}^1$.

Let us consider what the fibres of this family are. The point $x \in \mathbb{P}^1$ has preimage in the pull-back $\mathbb{P}^1 \times_{\{*\}} \mathbb{P}^1$ given by $\{x\} \times \mathbb{P}^1$, where x is fixed. The preimage is clearly isomorphic to $\mathbb{P}^1$. The sections $\sigma_1, \sigma_2, \sigma_3$ each identify the points $(x, 0), (x, 1), (x, \infty)$ on the fibre, while the new section δ identifies the point (x, x). By construction of $\mathfrak{X} \to \mathbb{P}^1$ as the stabilisation of this family, the fibre $\mathfrak{X}_x$ is the stabilisation of the curve $\{x\} \times \mathbb{P}^1$ with marks $(x, 0), (x, 1), (x, \infty), (x, x)$. If $x = 0, 1, \infty$, the marks coincide, so the stabilised family $\mathfrak{X}_x$ will have an additional twig. In contrast, if $x \neq 0, 1, \infty$, the fibre $\mathfrak{X}_x$ is formed of a single twig with four marks. See Figure 5 for a diagram showing how the structure of the fibre $\mathfrak{X}_x$ changes with different values of x.

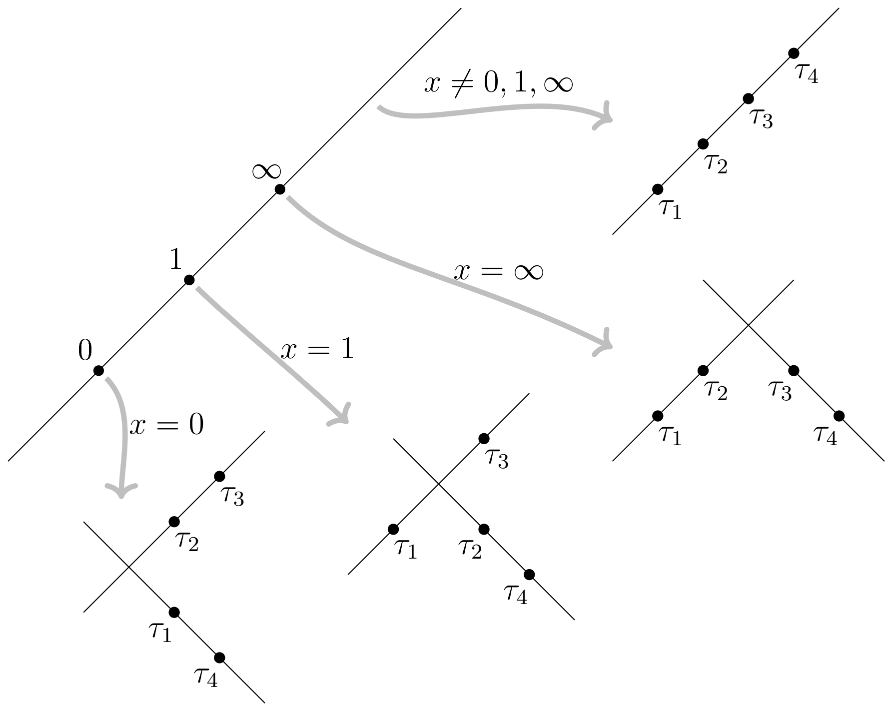

Figure 5: The fibre of $\mathfrak{X} \to \mathbb{P}^1$ at $x \in \mathbb{P}^1$ has two twigs if $x = 0, 1, \infty$, and a single twig otherwise. The distribution of the marks is different on each of the fibres $\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_\infty$, so these 4-pointed curves are not isomorphic.

The construction in Example 2.19 may be summarised in the following diagram.

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\phi"] \arrow[d, shift right] & \mathbb{P}^1 \times_{\{*\}} \mathbb{P}^1 \arrow[r] \arrow[d, shift right] & \mathbb{P}^1 \arrow[d, shift right, "\pi"'] \\
\mathbb{P}^1 \arrow[u, shift right, "\tau_1,\ldots,\tau_4"'] \arrow[r, equal] & \mathbb{P}^1 \arrow[u, shift right, "\delta"'] \arrow[r, "\pi"] & \{*\} \arrow[u, shift right, "\sigma_1,\sigma_2,\sigma_3"']
\end{tikzcd}$$

We will change our notation, using $\overline{M}_{0,3} = \{*\}$, $\overline{M}_{0,4} = \mathbb{P}^1$ and $\overline{M}_{0,5} = \mathfrak{X}$.

$$\begin{tikzcd}
\overline{M}_{0,5} \arrow[r, "\phi"] \arrow[d, shift right] & \overline{M}_{0,4} \times_{\overline{M}_{0,3}} \overline{M}_{0,4} \arrow[r] \arrow[d, shift right] & \overline{M}_{0,4} \arrow[d, shift right, "\pi"'] \\
\overline{M}_{0,4} \arrow[u, shift right, "\tau_1,\ldots,\tau_4"'] \arrow[r, equal] & \overline{M}_{0,4} \arrow[u, shift right, "\delta"'] \arrow[r, "\pi"] & \overline{M}_{0,3} \arrow[u, shift right, "\sigma_1,\sigma_2,\sigma_3"']
\end{tikzcd} \tag{7}$$

The diagram (7) correctly indicates how the argument in Example 2.19 may be generalised for $n \geq 3$. The next stage is summarised in (9).

Corollary 2.20. *The moduli space $\overline{M}_{0,n}$ is $(n-3)$-dimensional.*

Proof. By Theorem 2.17, $M_{0,n} \subset \overline{M}_{0,n}$ is an open subset. In (2), we saw how $M_{0,n}$ is given by an open subset of $\mathbb{P}^{n-3}$. Therefore $\overline{M}_{0,n}$ is $(n-3)$-dimensional as desired. $\square$

We started this section by observing that the isomorphism classes of ordered distinct quadruples in $\mathbb{P}^1$ are in bijective correspondence with $\mathbb{P}^1 - \{0, 1, \infty\}$. In Section 2.1, we were able to formally relate the geometric structure of this quasi-projective variety to the quadruples. Now, we have extended our definition of 4-pointed curves so that $\overline{M}_{0,4} = \mathbb{P}^1$, where the points $0, 1, \infty$ correspond to the 4-pointed curves with two twigs, as in Figure 5. In Section 4, we will see how well our extended definition relates the geometry of the moduli spaces to the geometry of the pointed curves.

3 Stable Maps

Recall from the Introduction that we were looking at ordered n-tuples on a curve because we wanted to look at degree-d maps that parametrise curves in $\mathbb{P}^r$. We were going to consider the maps, i.e. parametrisations, $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ with specified points of the source space $p_i \in \mathbb{P}^1$, and count those maps which satisfy $\mu(p_i) = q_i$, where q_i are points of $\mathbb{P}^r$ that we have fixed. Having now considered the moduli space of collections of points $p_i \in \mathbb{P}^1$ on a curve, it is time to start looking at the maps as above.

In Section 2.2, we extended the definition of a *curve* to a tree of projective lines in order to get a better moduli space. We will do the same here, without going through the motivational steps. The following definitions may be found in [4, Section 2.3].

Definition 3.1. An n-pointed map is a morphism $\mu : C \to \mathbb{P}^r$, where $C = (C; p_1, \dots, p_n)$ is a tree of projective lines with n distinct marks which are smooth points of C.

An isomorphism of n-pointed maps from $\mu : C \to \mathbb{P}^r$ to $\mu' : C' \to \mathbb{P}^r$ is an isomorphism $\phi : C \to C'$ which preserves the marks and $\mu = \mu' \circ \phi$.

As in Section 2.1, we will be working with families to count the isomorphism classes of the maps. This allows us to use the structure of the moduli space, rather than simply a bijection.

Definition 3.2. A family of n-pointed maps is a diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mu} & \mathbb{P}^r \\ {\scriptstyle \sigma_1,\dots,\sigma_n}\uparrow\downarrow{\scriptstyle \pi} & & \\ B & & \end{array}$$

where π is a flat family of trees of smooth rational curves and the sections σ_i are disjoint and do not meet the singularities of the fibers of π. Thus, given $b \in B$, the map μ restricted to the fiber gives an n-pointed map $\mu_b : \mathfrak{X}_b \to \mathbb{P}^r$ with marks $\sigma_i(b)$.

We must now restrict our notion of map and introduce the notion of *Kontsevich stability*. Since this is the only notion of the stability of a map we will use in this dissertation, we shall simply refer to the stability of a map.

Definition 3.3. An n-pointed map $\mu : C \to \mathbb{P}^r$ is stable if, and only if, any twig of C that is mapped under μ to a single point has at least three special points on it.

Let us compare the definitions of the stability of a curve (Section 2.2) and of the stability of a map. It is important to note that the source curve C of a stable map does not need to be itself stable as a curve. Indeed any map which is constant on no twig is stable, regardless of where the marks are placed. Thus, the stability of a curve becomes relevant precisely when and where a map is constant.

Example 3.4. Consider the family $\pi : \mathfrak{X} \to B$ of stable curves from Example 2.12 given by (3). In (4), we defined four disjoint sections σ_α for $\alpha = 0, 1, \infty, \Delta$. All four sections are required for the family of curves to be stable. In the following, we will look at different morphisms $\mu : \mathfrak{X} \to \mathbb{P}^r$ and see which sections σ_α are required for the family of *maps* to be stable.

Take the following two morphisms $\mu_r : \mathfrak{X} \to \mathbb{P}^r$ for $r = 1, 2$.

$$\mu_1(b, \mathbf{s}, \mathbf{x}) = \mathbf{s} \in \mathbb{P}^1 \qquad\qquad \mu_2(b, [s : t], \mathbf{x}) = [s^2 : s^2 : t^2] \in \mathbb{P}^2$$

A quick check of equations (5) and (6) will show that only the fibre $\mathfrak{X}_0$ at $b = 0$ has a twig that μ_r maps to a single point. We see that

$$\mu_1 : X \to \{[0 : 1]\} \in \mathbb{P}^1$$

so in order for the family with μ_1 to be stable, we must choose the sections σ_0 and σ_Δ to guarantee stability, for $\sigma_0(0), \sigma_\Delta(0) \in X$. Thus we can produce a family of stable 2-pointed maps $(\mathfrak{X} \to B; \sigma_0, \sigma_\Delta; \mu_1)$. We can include the other sections σ_1, σ_∞ to yield families of stable 3 and 4-pointed maps, but these sections are not required to ensure the stability of the family.

Similarly, there is only one twig mapped to a single point, namely $\mu_2 : X \to \{[0 : 0 : 1]\}$. Thus, we must again include the sections σ_0, σ_Δ in order for this family to be stable, for these provide marks on the twig X of the fibre $\mathfrak{X}_0$.

The motivation for stability is found in Theorem 3.5.

Theorem 3.5 ([4, Lemma 2.3.1])**.** *An n-pointed map is stable if, and only if, it has a finite number of automorphisms.*

The reader should follow the reference to see the proof. The forward direction involves looking at the map induced on the function fields. We will however demonstrate the proof of the reverse direction for the map $\mu_2|_{\mathfrak{X}_0}$ from Example 3.4, which captures the key idea of the argument.

Example 3.6. Let us use the notation from Figure 2, since we will only deal with the fibre $\mathfrak{X}_0$. We write $\mathfrak{X}_0 = S \cup X$, where $S, X \cong \mathbb{P}^1$ have coordinates

$[s : t]_S, [y : z]_X$ respectively, and where $v = [0 : 1]_S = [1 : 0]_X$ is the unique point lying on both twigs.

The sections σ_0, σ_Δ give us the marks $p_0 = [0 : 1]_X, p_\Delta = [1 : 1]_X$ on the twig X. Notice how any automorphism of X that preserves $v \in S \cap X$ extends to an automorphism of $\mathfrak{X}_0$. Define the automorphism $\psi_a : \mathfrak{X}_0 \to \mathfrak{X}_0$ on $\mathfrak{X}_0$ to be given by the identity on S and by $[y : z]_X \mapsto [y : az]_X$ on X. The reader will notice that ψ_a is compatible with $\mu_2|_{\mathfrak{X}_0}$, and that $\psi_a(p_0) = p_0$. Moreover, $\psi_a(p_\Delta) = [1 : 1/a]_X$, so for $a \neq 1$, ψ_a does not preserve the mark p_Δ. This means that if we fail to include the second mark p_Δ, we get an infinite family of automorphisms $\{\psi_a\}$.

A similar family may be produced when p_0 is not included. It is straightforward algebra to show that, in fact, the only non-trivial automorphism of $(\mathfrak{X}_0; p_0, p_\Delta; \mu_2|_{\mathfrak{X}_0})$ is $[s : t]_S \mapsto [s : -t]_S$ on S and the identity on X.

3.1 The coarse moduli space

We would like to construct a fine moduli space for stable n-pointed maps, however this is not possible. The existence of non-trivial automorphisms, such as the one we found in Example 3.6, means that no such moduli space could exist. The relationship between automorphisms and moduli spaces is not quite as simple as the previous sentence suggests, and [3, Example 2.1] provides a neat example as evidence.

Nonetheless, we will have to generalise the concept of moduli space for us to be able to use, or even find, any structure on the space of isomorphism classes of stable n-pointed maps.

Before, in Definition 2.5, we had a family U/M with a universal pull-back property. Our new weaker definition will use the object M and the pull-back property, and we will discard U. The pull-back property, of course, does not make sense without U, so it is converted into the following concept.

Definition 3.7. [1, Example 5.2] A morphism between families $\mathfrak{X}/B$ and $\mathfrak{X}'/B'$ of stable n-pointed curves is a pair of morphisms $\mathfrak{X} \to \mathfrak{X}'$ and $B \to B'$ such that

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{X}' \\ {\scriptstyle \sigma_1,\ldots,\sigma_n}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle \sigma'_1,\ldots,\sigma'_n} \\ B & \longrightarrow & B' \end{array}$$

commutes, and moreover that $\mathfrak{X}$ is a pull-back of $\mathfrak{X}'/B'$ along the morphism $B \to B'$.

Let $M = \overline{M}_{0,n}$ be the fine moduli space from Theorem 2.6. We will isolate the properties of M that form the weaker definition of moduli space.

Given any family $\mathfrak{X}/B$, there is a unique morphism $B \to M$ such that $\mathfrak{X}/B$ is induced as the pull-back along this morphism. Denote this morphism $\Psi_M(\mathfrak{X}/B) : B \to M$.

Proposition 3.8. The fine moduli space $M = \overline{M}_{0,n}$ satisfies the following properties.

1. The assignment Ψ_M is compatible with morphisms of families, so the following diagram commutes.

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[d] & & \mathfrak{X}' \arrow[d] \\ B \arrow[rr] \arrow[dr, "\Psi_M(\mathfrak{X}/B)"'] & & B' \arrow[dl, "\Psi_M(\mathfrak{X}'/B')"] \\ & M & \end{tikzcd}$$

   That is, Ψ_M is a functor

$$\{\text{families of stable } n\text{-pointed curves}\} \longrightarrow \operatorname{Hom}(\cdot, M)$$

2. The functor Ψ_M induces a bijection

$$\frac{\{C/\{*\}\}}{\text{isomorphisms}} \longleftrightarrow \operatorname{Hom}(\{*\}, M)$$

3. (Universal property) Suppose N is any quasi-projective variety with another assignment of morphisms $\Psi_N(\mathfrak{X}/B) : B \to N$, which is compatible with morphisms of families so as to become a functor. Then there exists a unique morphism $\varphi : M \to N$ such that the following diagram commutes for all families $\mathfrak{X}/B$.

$$\begin{tikzcd} \mathfrak{X} \arrow[d] & & \\ B \arrow[rr, "\Psi_M(\mathfrak{X}/B)"] \arrow[dr, "\Psi_N(\mathfrak{X}/B)"'] & & M \arrow[dl, dashed, "\varphi"] \\ & N & \end{tikzcd} \tag{8}$$

Proof. 1. Given such a morphism between two families, the family $\mathfrak{X}/B$ is the pull-back of the family $\mathfrak{X}'/B'$ along $B \to B'$. By definition of M, the family $\mathfrak{X}'/B'$ is the pull-back of the universal family along $\Phi_M(\mathfrak{X}'/B')$. It follows that $\mathfrak{X}/B$ is the pull-back of the universal family along the composition $B \to B' \to M$, and hence, by uniqueness, that this composition is $\Phi_M(\mathfrak{X}/B)$.

2. The inverse assignments are the pull-backs of the morphisms $\{*\} \to M$.

3. We are given a canonical morphism $\Psi_N(\overline{U}_{0,n}/M) : M \to N$. Since Ψ_N is compatible with morphisms, and in particular with pull-back morphisms, diagram (8) is commutative for this choice of morphism $M \to N$, showing existence. The uniqueness claim follows from the commutativity of the following diagram, which is (8) applied to the universal family.

$$\begin{tikzcd}
\overline{U}_{0,n} \arrow[d] & & \\
M \arrow[rr, "\mathrm{Id}_M"] \arrow[dr, "\Psi_N(\overline{U}_{0,n}/M)"'] & & M \arrow[dl, dashed, "\varphi"] \\
& N &
\end{tikzcd}$$

□

This leads us to define a coarse moduli space for families of maps by adapting the statements of Proposition 3.8 to this new setting. The definitions of pull-backs and of morphisms translate easily to stable maps. A reader who is interested instead in the category-theoretic definitions of fine and coarse moduli spaces should seek [3, Chapter 2].

Definition 3.9. A coarse moduli space of stable n-pointed maps to $\mathbb{P}^r$ of degree d is a quasi-projective variety $M = \overline{M}_{0,n}(\mathbb{P}^r, d)$ with an associated assignment Ψ_M satisfying the following properties. To every family of maps $(\mathfrak{X}/B; \mu)$, we assign a morphism

$$\Psi_M(\mathfrak{X}/B; \mu) : B \to M$$

1. This assignment is compatible with morphisms of families, so that the following diagram is commutative. That is, Ψ_M is a functor between appropriate categories.

$$\begin{tikzcd}
& \mathbb{P}^r & \\
\mathfrak{X} \arrow[ur, "\mu"] \arrow[rr] \arrow[d] & & \mathfrak{X}' \arrow[ul, "\mu'"'] \arrow[d] \\
B \arrow[rr] \arrow[dr, "\Psi_M(\mathfrak{X}/B)"'] & & B' \arrow[dl, "\Psi_M(\mathfrak{X}'/B')"] \\
& M &
\end{tikzcd}$$

2. The functor Ψ_M induces a bijection

$$\frac{\{(C/\{*\};\mu)\}}{\text{isomorphisms}} \longleftrightarrow \mathrm{Hom}\left(\{*\}, M\right)$$

3. (Universal property) Given any other quasi-projective variety N and an assignment Ψ_N compatible with morphisms, there is a unique morphism $\varphi : M \to N$ such that the following diagram commutes for all families $(\mathfrak{X}/B;\mu)$.

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\mu"] \arrow[d] & \mathbb{P}^r & \\
B \arrow[rr, "\Psi_M(\mathfrak{X}/B;\mu)"] \arrow[dr, "\Psi_N(\mathfrak{X}/B;\mu)"'] & & M \arrow[dl, dashed, "\varphi"] \\
& N &
\end{tikzcd}$$

Part 2 of Definition 3.9 ensures that M indeed corresponds to the isomorphism classes, as desired. Parts 1 and 3 correspond to giving the correct structure to M, and this is seen in the way they are each more abstract and *natural*.

Unsurprisingly, this definition will be followed by the existence theorem for the coarse moduli space.

Theorem 3.10. [4, Theorems 2.3.2, 2.3.3] *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ parametrising the isomorphism classes of stable n-pointed maps to $\mathbb{P}^r$ of degree d. $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a projective normal irreducible variety.*

3.2 Evaluation maps

Since morphisms of stable maps preserve their marks, we can define the following important maps.

Definition 3.11. For each i, there is a natural map

$$\begin{aligned} \nu_i : \overline{M}_{0,n}(\mathbb{P}^r, d) &\to \mathbb{P}^r \\ (C; p_1, \ldots, p_n; \mu) &\mapsto \mu(p_i) \end{aligned}$$

called the i-th evaluation map.

In the above definition, we have implicitly used part 2 of Definition 3.9 to define ν_i. For this to work, we require that ν_i is well-defined on isomorphism classes of stable maps. This will follow from Proposition 3.12.

Proposition 3.12. The evaluation maps are morphisms of quasi-projective varieties.

Proof. We will use the universal property of the coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$. To any family $(\mathfrak{X}/B; \mu)$, assign the morphism $\mu \circ \sigma_i : B \to \mathbb{P}^r$. Since marks are preserved by morphisms, this assignment is compatible with morphisms, as is shown in the following commutative diagram.

$$\begin{tikzcd}
\mathbb{P}^r \arrow[r, equal] & \mathbb{P}^r \\
\mathfrak{X} \arrow[u, "\mu"] \arrow[r] & \mathfrak{X}' \arrow[u, "\mu'"'] \\
B \arrow[u, "\sigma_i"] \arrow[r] & B' \arrow[u, "\sigma_i'"']
\end{tikzcd}$$

The universal property yields a morphism $\nu_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$. Finally, since our assignment gives the desired values on stable maps (over a point), so too will the morphism ν_i on the isomorphism classes. $\square$

3.3 Forgetful morphism

Much like the forgetful morphism $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ from Section 2.5, whose existence was guaranteed by Theorem 2.13, we have a forgetful morphism $\varepsilon : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$ for $n \geq 3$. Given any stable n-pointed map $\mu : C \to \mathbb{P}^r$ in $\overline{M}_{0,n}(\mathbb{P}^r, d)$, losing the map $\mu : C \to \mathbb{P}^r$ and keeping the tree of projective lines C with the marks $p_1, \ldots, p_n \in C$ will yield an n-pointed curve. By contracting twigs with fewer than three special points, we will get a uniquely defined stable n-pointed curve in $\overline{M}_{0,n}$. We have again used part 2 of Definition 3.9 to define our map ε. We have described ε as a map between sets, but in fact ε is a morphism of varieties.

3.4 Dimension

In this section, we will provide an informal argument for Proposition 3.13. Many of the arguments are incomplete, and there are references for the claims that are provided with no proof at all. This is a condensed version of the reasoning provided by Kock and Vainsencher in [4, Section 2].

Proposition 3.13. [4, 2.3.4] The dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is

$$\begin{aligned}
\dim \overline{M}_{0,n}(\mathbb{P}^r, d) &= (r+1)(d+1) - 1 - 3 + n \\
&= rd + r + d + n - 3
\end{aligned}$$

We start by considering the maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d. Recall the degree is defined as in Definition 1.1, and, as stated in Section 1.1, degree-d maps are parametrised by $r+1$ binary forms of degree d, such that they do not all simultaneously vanish anywhere, and up to constant factor [4, 2.2.1]. This defines a subset

$$W(r,d) \subset \mathbb{P}(\bigoplus_{i=0}^{r} \bigoplus_{j=0}^{d} \mathbb{C}x^j y^{d-j})$$

The dimension of the ambient space is $(r+1)(d+1)-1$, for we are in the projective space of a $(r+1)(d+1)$-dimensional vector space.

It is a fact that $W(r,d)$ is a Zariski open subset [4, 2.2.1], and hence we get that $\dim W(r,d) = (r+1)(d+1)-1$.

We can construct a family of maps (without any marks) over $W(r,d)$, where on the fibre of $f \in W(r,d)$, we set $\mu = \mu_f : \mathbb{P}^1 \to \mathbb{P}^r$ to be given by the evaluation $f(x,y)$ of the binary form at the point of $[x:y] \in \mathbb{P}^1$.

$$\begin{tikzcd} W(r,d) \times \mathbb{P}^1 \arrow[r, "\mu"] \arrow[d] & \mathbb{P}^r \\ W(r,d) & \end{tikzcd}$$

Since no binary form can induce a constant map, this is automatically a family of stable 0-pointed maps. Therefore there exists a map $\varphi : W(r,d) \to \overline{M}_{0,0}(\mathbb{P}^r, d)$ by the definition of the coarse moduli space. Moreover, for $f, g \in W(r,d)$, we have $\varphi(f) = \varphi(g)$ if, and only if, the binary forms f, g induce isomorphic maps on their fibres.

Suppose the map μ_f has no automorphisms. That is, $\operatorname{Aut}(\mu_f)$ is trivial. Then the preimage $\varphi^{-1}(f)$ will consist precisely of the different representations of μ_f under different coordinates. This suggests $\varphi^{-1}(f) \cong \operatorname{Aut}(\mathbb{P}^1)$, and in particular that $\dim \varphi^{-1}(f) = 3$.

It turns out that there is an open subset of $W(r,d)$ which consists of automorphism-free binary forms [4, Proposition 2.1.8 and Lemma 2.1.14], and this gives us [4, 2.1.16]

$$\begin{aligned} \dim \overline{M}_{0,0}(\mathbb{P}^r, d) &= \dim W(r,d) - \dim \operatorname{Aut}\left(\mathbb{P}^1\right) \\ &= (r+1)(d+1) - 1 - 3 \end{aligned}$$

where we have used that φ surjects onto an open subset of $\overline{M}_{0,0}(\mathbb{P}^r, d)$.

Finally, the subset $M_{0,n}(\mathbb{P}^r, d)$ of maps whose source space has only one twig is open in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ [2, Section 0.4]. On this subset, we can see that choosing the position of the marks is a free choice, subject to avoiding the other marks, and thus each mark should correspond to an additional dimension. In this way, we get the desired result [4, 2.3.4] as

$$\begin{aligned}\dim \overline{M}_{0,n}(\mathbb{P}^r, d) &= \dim \overline{M}_{0,0}(\mathbb{P}^r, d) + n \\ &= (r+1)(d+1) - 1 - 3 + n\end{aligned}$$

4 The Boundary

We have now seen the moduli spaces $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(\mathbb{P}^r, d)$. We took some time to find a way to extend our definition of curve so that we get a projective variety, rather than a quasi-projective variety (c.f. Section 2.2). It is now that we see how this is useful.

From Theorem 2.17, we know that $M_{0,n}$ is an open dense subset of $\overline{M}_{0,n}$. In this section, we look at the structure of the boundary $\overline{M}_{0,n} \setminus M_{0,n}$. It is immediate from definitions that any point in the boundary corresponds to a reducible curve; that is, the fibre of the point is a curve with more than one twig.

Example 4.1. We will look at the points of $\overline{M}_{0,5}$ which correspond to curves which are made of two twigs, one with marks $\sigma_1, \sigma_2, \sigma_3$ and the other twig with marks σ_4, σ_5, as shown in Figure 6*(A)*. I will denote this subset $D(\{1,2,3\},\{4,5\}) \subset \overline{M}_{0,5}$.

Figure 6*(B)* contains three curve configurations with two nodes, and these occur when two of the marks $\sigma_1, \sigma_2, \sigma_3$ would have coincided. Thus these three additional curves are really just special cases, and they should correspond to points in $D(\{1,2,3\},\{4,5\})$.

Thus informally, $D(\{1,2,3\},\{4,5\})$ contains all curves that are in Figure 6. We can, in fact, write down a concrete description of these curves as follows.

Consider the following diagram, which is a repeat of diagram (7) for the second case of recursion when constructing the moduli spaces. Notice that (9) may be concatenated with (7).

$$\begin{tikzcd}
\overline{M}_{0,6} \arrow[r] \arrow[d] & \overline{M}_{0,5} \times_{\overline{M}_{0,4}} \overline{M}_{0,5} \arrow[r] \arrow[d, shift right] & \overline{M}_{0,5} \arrow[d, shift right, "\pi"'] \\
\overline{M}_{0,5} \arrow[r, equal] & \overline{M}_{0,5} \arrow[r] \arrow[u, shift right, "\delta"'] & \overline{M}_{0,4} \arrow[u, shift right, "\sigma_1,\sigma_2,\sigma_3,\sigma_4"']
\end{tikzcd} \tag{9}$$

The set $D(\{1,2,3\},\{4,5\})$ contains those points at which the fifth section δ coincided with σ_4, before stabilisation. Thus

$$\begin{aligned} D(\{1,2,3\},\{4,5\}) &= \{x \in \overline{M}_{0,5} \mid \delta(x) = \sigma_4(x)\} \\ &= \{x \in \overline{M}_{0,5} \mid x = (\sigma_4 \circ \pi)(x)\} \end{aligned} \tag{10}$$

The advantage of (10) is that we can easily see that $D(\{1,2,3\},\{4,5\})$ is a closed subset of $\overline{M}_{0,5}$, and thus is a projective variety.

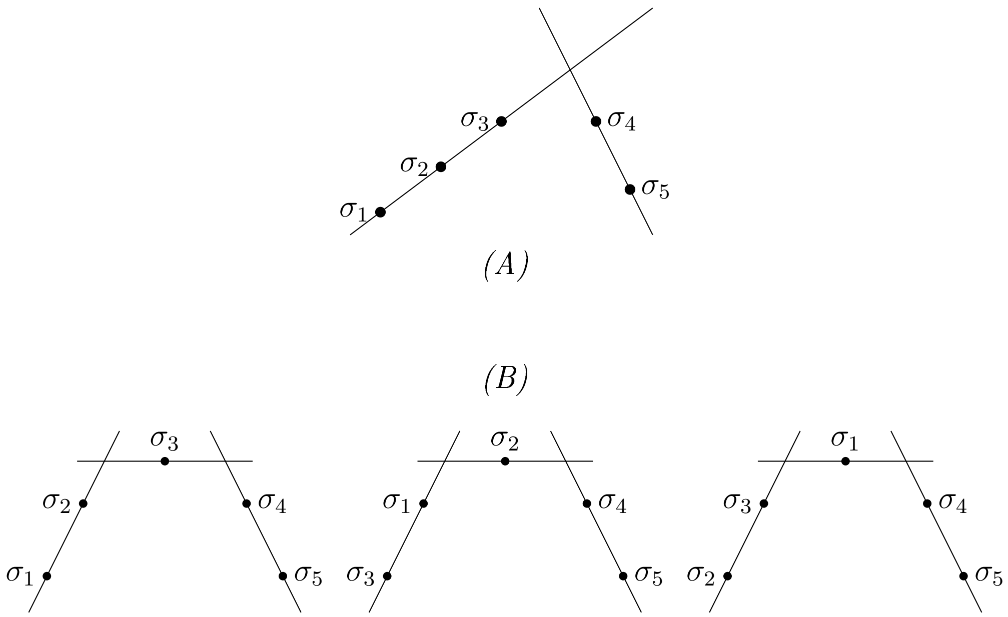

Figure 6: *(A)* describes a configuration of stable 5-pointed curves with one node. *(B)* describes three configurations of stable 5-pointed curves with two nodes. Each of the configurations in *(B)* corresponds to a unique curve in $\overline{M}_{0,5}$. The curves in *(B)* are obtained by letting marks on the left twig of *(A)* converge, and so should be treated as special cases of *(A)*.

Definition 4.2. [4, 1.5.4] Denote by $[n]$ the set of n marks of a curve. Let A, B be disjoint subsets of $[n]$ with $[n] = A \cup B$, and each with at least two elements. The boundary divisor $D(A, B)$ is the irreducible projective subvariety of $\overline{M}_{0,n}$ whose general elements correspond to curves with two twigs, with the marks of A on one twig and the marks of B on the other.

Thus, in Example 4.1, the general points correspond to those curves that look like Figure 6*(A)*. The points that correspond to the curves in Figure 6*(B)* are required in order that $D(\{1, 2, 3\}, \{4, 5\})$ is a projective variety, though we did not show this. All boundary divisors may be defined using the recursive structure of $\overline{M}_{0,n}$ from Example 2.19 to give an explicit formula like (10), and it is an instructive exercise to find the explicit formula for a few examples.

Throughout, we will assume that A, B satisfy the conditions in Definition 4.2. We will define a divisor later, and we will see that a boundary divisor is indeed a divisor. First, we will state and provide a partial proof of Theorem 4.3, which is fundamental to understanding the boundary.

Theorem 4.3. [4, 1.5.10] *Given A, B subsets of $[n]$, we have a canonical* *isomorphism*

$$D(A,B) \cong \overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$$

where $\overline{M}_{0,A\cup\{x\}}$ *is the moduli space of stable curves with marks labeled by elements of* $A \cup \{x\}$*, and* $\overline{M}_{0,B\cup\{x\}}$ *with marks* $B \cup \{x\}$*.*

Proof. [4, 1.5.10] The key insight in this proof is shown in Figure 7, where we see how each of the moduli spaces in the product encode the information of the curve in the boundary divisor. In particular, the location of the node of the reducible curve corresponds to the location of the sections x.

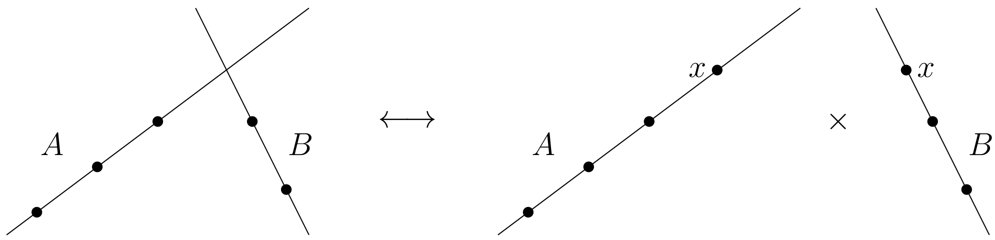

Figure 7: [4, 1.5.10] Boundary curves in $D(A,B)$ correspond to curves in the product $\overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$, and vice versa.

Take the family over $D(A,B)$, and forget all but one of the marks of B. Denote the remaining mark of B by x. Notice that x is really just the intersection point of the two twigs now. This gives us a family of stable curves with $|A\cup\{x\}|$ marks. Thus by the universal property, we get a unique morphism $D(A,B) \to \overline{M}_{0,A\cup\{x\}}$.

We similarly get a canonical morphism $D(A,B) \to \overline{M}_{0,B\cup\{x\}}$, and can use this to get a unique morphism to the product $\overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$.

Conversely, a point of the product $\overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$ gives an n-pointed curve, by joining the two curves at the points marked x with a node. Going through the precise construction of how to get a new curve by identifying the two points at a node is beyond the scope of this dissertation, so we will assume that this operation is possible, and that it is possible to do so in a family. Thus we get a family of n-pointed curves over the product $\overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$, and hence a morphism $\overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}} \to \overline{M}_{0,n}$, which by construction lands in $D(A,B)$.

One can verify that both compositions of these morphisms give the identity, hence we have found the desired isomorphism. □

Example 4.4. Take $D(\{1,2,3\},\{4,5\})$ from Example 4.1. By Theorem 4.3, there is a canonical isomorphism

$$\begin{aligned} D(\{1,2,3\},\{4,5\}) &\cong \overline{M}_{0,4} \times \overline{M}_{0,3} \\ &\cong \mathbb{P}^1 \end{aligned}$$

Notice that we had to add three *degenerate* curves in Figure 6*(B)* to get a projective variety, much like how we added three degenerate curves to $M_{0,4}$ to get the projective variety $\overline{M}_{0,4}$. These two sets of three degenerate curves correspond to each other via the isomorphism.

Corollary 4.5. *The projective variety $D(A,B) \subset \overline{M}_{0,n}$ has dimension $n-4$.*

Proof. Recall $\dim \overline{M}_{0,n} = n-3$ from Corollary 2.20. Then

$$\begin{aligned}
\dim D(A,B) &= \dim \left(\overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}\right) \\
&= \dim \overline{M}_{0,A\cup\{x\}} + \dim \overline{M}_{0,B\cup\{x\}} \\
&= |A \cup \{x\}| - 3 + |B \cup \{x\}| - 3 \\
&= |A| + |B| - 4 \\
&= n - 4
\end{aligned}$$

□

Consider the following two examples of intersections of boundary divisors.

Example 4.6. In $\overline{M}_{0,5}$, the intersection

$$D(\{1,2,3\},\{4,5\}) \cap D(\{1,2\},\{3,4,5\})$$

is precisely the leftmost curve in Figure 6*(B)*, whereas the intersection

$$D(\{1,2,3\},\{4,5\}) \cap D(\{1,5\},\{2,3,4\})$$

is empty.

The results of Example 4.6 hold true in general, in that the intersection of two boundary divisors is either a codimension-2 subvariety or empty [4, Remark 1.5.7]. Further intersections continue in a similar fashion, with each additional node contributing a further codimension. This leads to a stratification of the boundary [4, Example 1.5.2]. We do not pursue this idea further in this dissertation.

4.1 Weil Divisors

We are now in a position to define a divisor, and to see that $D(A,B)$ is indeed a divisor as we claimed earlier. There are, in fact, three relevant definitions or notions of divisors. These are Weil divisors, Cartier divisors and line bundles or invertible sheaves with a rational section. In the context of a smooth variety, these concepts are equivalent, and so they are the same for $\overline{M}_{0,n}$.

We will define and work exclusively with the simpler of the definitions, that of the Weil divisor. As a result, there will be moments where we will quote properties that follow from the other definitions, and it will be less intuitive why these are true. We will be explicit when we use these properties.

Definition 4.7. [6, Section 14.2] Let X be a smooth projective variety. A Weil divisor is a formal sum of irreducible codimension 1 subvarieties of X

$$D = \sum_{Y \subset X} n_Y Y$$

where $n_Y \in \mathbb{Z}$, with finitely many n_Y nonzero.

The Weil divisors form an Abelian group $\mathrm{Weil}\,(X)$. We showed in Corollary 4.5 that $D(A, B)$ has codimension 1, and I claimed that it is irreducible, hence $D(A, B)$ is one of the possible summands of a divisor, and is itself a divisor.

Definition 4.8. The support of a divisor $\sum n_Y Y$ is $\cup\{Y \mid n_Y \neq 0\}$.

We would like to consider *linearly equivalent* divisors. We will (informally) define a subgroup of *principal divisors*, and two divisors shall be deemed linearly equivalent if they differ by a principal divisor. We will denote the linear equivalence of divisors as $D \sim D'$, and the quotient group as $\mathrm{Cl}\,(X) = \mathrm{Weil}\,(X) / \sim$, which is called the *divisor class group*.

Take any rational function f on X which has a reciprocal in $K(X)$. There are *valuations* $v_Y(\cdot)$ which give integers corresponding to the following intuition. If $v_Y(f) > 0$, then f has a zero of order $v_Y(f)$ on Y, and if $v_Y(f) < 0$, then f has a pole of order $v_Y(f)$ on Y [5, Lecture 1]. For $\mathbb{P}^1$, this intuition is exactly correct, where $Y = \{p\}$ is a point of $\mathbb{P}^1$ and $v_Y(f)$ is the order of the zero or pole of the meromorphic function f at p. The principal divisor of f is

$$(f) = \sum_{Y \subset X} v_Y(f) \cdot Y$$

It is a fact that $K(X)^\times \to \mathrm{Weil}\,(X)$ is a group homomorphism, so that the principal divisors form a subgroup as desired [5, Lemma 1.5].

Example 4.9. Let $p = [p_1 : p_2], q = [q_1 : q_2]$ be any two distinct points in $\mathbb{P}^1$. Define the meromorphic function $f : \mathbb{P}^1 \setminus \{q\} \to \mathbb{C}$ by

$$f([x : y]) = \frac{p_2 x - p_1 y}{q_2 x - q_1 y}$$

Then f has a zero at p and a pole at q, each of order 1. It follows that $p - q$ is a principal divisor.

Example 4.10. Now $\overline{M}_{0,4} \cong \mathbb{P}^1$ and $D(A, B)$ corresponds to the points $0, 1, \infty$ under the isomorphism, for the different choices of A and B. Thus we have just found that

$$D(\{1,2\},\{3,4\}) \sim D(\{1,3\},\{2,4\}) \sim D(\{1,4\},\{2,3\}) \tag{11}$$

This equivalence is the backbone of the proof of Kontsevich's formula.

Example 4.11. Indeed, the degree map $\deg : \mathrm{Cl}(\mathbb{P}^1) \to \mathbb{Z}$ is a group isomorphism, where

$$\deg\left(\sum_{p\in\mathbb{P}^1} n_p \cdot p\right) = \sum_{p\in\mathbb{P}^1} n_p$$

It is not hard to check that deg is a well-defined epimorphism. This is a consequence of meromorphic functions having the same number of zeros as poles. The map is injective because any two divisors with the same degree are linearly equivalent. This follows from Example 4.9 and the transitivity of $\sim$.

The final concept we need is the pull-back of a divisor. Unfortunately, we cannot appreciate the contents of what the pull-back of a divisor is without the other definitions of divisor. Consequently, we will not define what the pull-back of a divisor is in general, and will instead only look at the case that interests us. We will adapt [4, 1.5.11] so that it becomes our definition of a pull-back, rather than the argument provided there. The intuition provided below is similar to Kock and Vainsencher's argument.

The morphism we want to pull back is the forgetful morphism $\varepsilon : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$. This morphism can be thought of either as the projection map in the universal family of n-pointed curves, or as the map forgetting the last mark and stabilising. Due to the recursive structure of the moduli spaces, these are easily seen to be equivalent.

Let the mark we are forgetting be q, and let $D(A, B)$ be a boundary divisor of $\overline{M}_{0,n}$. We will first consider what the preimage of $D(A, B)$ is. On the general curve in $D(A, B)$, the point q could either lie on the A-marked twig, or on the B-marked twig. This motivates the claim that

$$\varepsilon^{-1}D(A, B) = D(A \cup \{q\}, B) \cup D(A, B \cup \{q\})$$

We can check that this also matches our intuition of what happens for the curves with more than two twigs. For example, when q is on the intersection of the two twigs of the general curve, then the preimage is the curve with three twigs, the middle of which contains two nodes and q.

We will define the pull-back of $D(A,B)$ to be

$$\varepsilon^* D(A,B) = D(A \cup \{q\}, B) + D(A, B \cup \{q\})$$

Notice the support of the pull-back divisor is the preimage we found.

We need the following important fact about pulling back divisors.

Proposition 4.12. If $D \sim D'$ are linearly equivalent, then the pull-back divisors $\varepsilon^* D \sim \varepsilon^* D'$ are also linearly equivalent.

Combining the results of Section 4 gives us the following theorem.

Theorem 4.13. [4, 1.5.14] *Let $n \geq 4$. Label four of the marks of $\overline{M}_{0,n}$ as i, j, k, l. Then*

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A,B) \sim \sum_{\substack{i,k \in A \\ j,l \in B}} D(A,B) \sim \sum_{\substack{i,l \in A \\ j,k \in B}} D(A,B)$$

Proof. Notice that

$$\sum_{\substack{1,2 \in A \\ 3,4 \in B}} D(A,B) = \varepsilon_n^* \cdots \varepsilon_5^* D(\{1,2\},\{3,4\})$$

is the composition of pull-backs of the morphisms that forget each of the other marks. This can be shown by induction on n. In Example 4.10, we showed that the divisors in $\overline{M}_{0,4}$ are linearly equivalent. Then Proposition 4.12 gives the desired linear equivalences. $\square$

4.2 The Boundary of the Moduli Space of Maps

The above work with the boundary of $\overline{M}_{0,n}$ can be repeated for $\overline{M}_{0,n}(\mathbb{P}^r, d)$. We will quickly summarise the main results given in [4, Section 2.7], and our exposition of them will not differ hugely from Kock and Vainsencher's work.

Be warned that $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is not a smooth variety, so we do not have an equivalence between the different definitions of divisor. Any Cartier divisor of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a Weil divisor however [5, Lecture 2, Definition 2.3], so we can still state the following results using Weil divisors. Later, we will be restricting our attention to a smooth subset of $\overline{M}_{0,n}(\mathbb{P}^r, d)$, so this no longer is a problem.

Definition 4.14. [4, 2.7.1] Let $A, B \subset [n]$ with $A \cup B = [n]$ and $d_A + d_B = d$, and assume $|A| \geq 2$ if $d_A = 0$, and $|B| \geq 2$ if $d_B = 0$. Let $D(A, B; d_A, d_B)$ be the subvariety whose general points correspond to maps μ with a domain $C = C_A \cup C_B$ with two twigs, where the marks of A lie on C_A and the marks of B lie on C_B, and where the restriction $\mu|_{C_A}$ is a map of degree d_A and $\mu|_{C_B}$ is of degree d_B.

Proposition 4.15. [2, 6.1] The subvariety $D(A, B; d_A, d_B) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ is irreducible and of codimension 1, so that it is an (irreducible) Weil divisor.

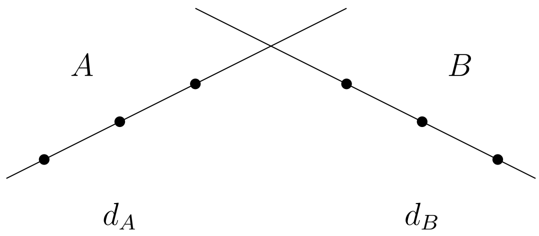

Figure 8: [4, 2.7.1] The boundary divisor $D(A, B; d_A, d_B) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ will be drawn as above, with the marks of A labeling the marks on the left twig, and the degree of $\mu|_A$ the map restricted to the A-twig drawn below the twig, and likewise for B.

Proposition 4.16. ([2, 6.2] and [4, 2.7.4]) Suppose that $A, B \neq \emptyset$. Then $D(A, B; d_A, d_B)$ is canonically isomorphic to the fibre product of

$$\begin{tikzcd} & \overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \arrow[d, "\nu_x"] \\ \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) \arrow[r, "\nu_x"'] & \mathbb{P}^r \end{tikzcd}$$

As in Theorem 4.3, this isomorphism arises as the gluing map, where the domains are glued at the additional mark x. We need $\mu(x) \in \mathbb{P}^r$ to be well-defined, hence taking the fibre product, rather than the ordinary product.

Definition 4.17. Let $n \geq 4$. The pull-back along the forgetful morphism $\varepsilon : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$ from Section 3.3 of the divisor $D(A, B) \subset \overline{M}_{0,n}$ is given by

$$\varepsilon^* D(A, B) = \sum_{d_A + d_B = d} D(A, B; d_A, d_B)$$

Notice that $|A|, |B| \geq 2$ is implied because we are considering a boundary divisor of $\overline{M}_{0,n}$. Notice also that the support of the pull-back divisor is the preimage $\varepsilon^{-1} D(A, B)$.

Theorem 4.18. [4, 2.7.6] *Let $n \geq 4$. Let i, j, k, l be four of the marks of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. The pull-back divisor of $D(\{i, j\}, \{k, l\}) \subset \overline{M}_{0,4}$ along the composition of forgetful morphisms $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n} \to \overline{M}_{0,4}$ is given by*

$$D(ij, kl) = \sum_{\substack{i,j \in A \\ k,l \in B \\ d_A + d_B = d}} D(A, B; d_A, d_B)$$

Then Proposition 4.12 and Theorem 4.13 give us

$$D(ij, kl) \sim D(ik, jl) \sim D(il, jk)$$

5 Kontsevich's Formula

We are now in a position to sketch the proof of the formula. We will be following the proof provided in [4, Sections 3.2 and 3.3] very closely, however we will refer to what we have shown or stated in this dissertation, as opposed to the wider range of results available to Kock and Vainsencher.

Theorem (Kontsevich)**.** [4, Theorem 3.3.1] *Let N_d be the number of rational curves of degree d passing through $3d-1$ general points in the plane $\mathbb{P}^2$. Then $N_1 = 1$ and for all d,*

$$\begin{aligned} N_d \; + \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \tbinom{3d-4}{3d_A-1} \cdot d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_B \\ = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \tbinom{3d-4}{3d_A-2} \cdot d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B \end{aligned} \tag{1}$$

Proof. [4, Proposition 3.3.1] Take two lines L_1, L_2 in $\mathbb{P}^2$ and $3d-1$ points $Q_1, \dots, Q_{3d-2} \in \mathbb{P}^2$ in general position. By this, we mean

1. $L_1 \neq L_2$
2. $Q_i \notin L_j$
3. Of the points $Q_1, \dots, Q_{3d-2}, L_1 \cap L_2$,
   - No three are colinear.
   - No six lie on a conic.
   - And, in general, no $3e$ such points lie on a degree-e curve, for $e < d$.

The points $Q_1, \dots, Q_{3d-2}, L_1 \cap L_2$ will be the $3d-1$ general points in the statement of the theorem.

Let $n = 3d$. We will be working in $\overline{M}_{0,n}(\mathbb{P}^2, d)$, and we will label the marks $m_1, m_2, p_1, \dots, p_{n-2}$. Define the subset $Y \subset \overline{M}_{0,n}(\mathbb{P}^2, d)$ to be those stable maps which map p_i to Q_i and m_j into L_j, using the part 2 of Definition 3.9. We can write Y as an intersection of preimages of the evaluation maps from Proposition 3.12

$$Y = \nu_{m_1}^{-1}(L_1) \cap \nu_{m_2}^{-1}(L_2) \cap \nu_{p_1}^{-1}(Q_1) \cap \cdots \cap \nu_{p_{n-2}}^{-1}(Q_{n-2})$$

so Y is a subvariety of $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

In fact, Y is a curve. Kock and Vainsencher argue that the preimage of a line in $\mathbb{P}^2$ has codimension 1 and the preimage of a point has codimension 2 using the flatness of the evaluation maps. They then use the generality of the chosen points and lines to show that each intersection contributes its codimension, so that Y is of codimension $2(n-2)+1+1 = 6d-2$. By Proposition 3.13, the ambient space has dimension $3d+n-1 = 6d-1$, which gives that Y is indeed a curve. For example, in the $d=2$ case, we are working in $\overline{M}_{0,6}(\mathbb{P}^r, 2)$ of dimension 11, and Y has codimension 10.

In the proof, we will be looking at the intersection of Y and the boundary divisors. According to Kock and Vainsencher, the intersection with each boundary divisor occurs in a smooth open subset $\overline{M}^*_{0,n}(\mathbb{P}^2, d) \subset \overline{M}_{0,n}(\mathbb{P}^2, d)$. The subset $\overline{M}^*_{0,n}(\mathbb{P}^2, d)$ is interesting in its own right, and is, for example, a fine moduli space for a restricted category of stable maps [2, Theorem 2 (iii)]. It is relevant to us, however, because now Weil divisors and Cartier divisors are equivalent concepts when discussing the intersection. It is the fact we are working in this subset that also guarantees the maps μ in the intersection are birational onto their image, and hence that we are counting the correct objects. A full justification may be found in [4, Sections 3.4 and 3.5].

Kock and Vainsencher also argue that the generality of the points and lines implies the intersection with each boundary divisor is a transverse intersection. This translates to the fact that the intersections $Y \cap D(A, B; d_A, d_B)$ are divisors in Y. Since Y is a curve, this means that the intersection is a finite collection of points, and the corresponding divisor in Y is the sum of those points. It is intuitive that if two divisors are linearly equivalent in the larger space X, then when restricted to a smaller space $Y \subset X$, they remain linearly equivalent. The intuition is that any principal divisor (f) in X should become a principal divisor $(f|_Y)$ in Y.

We have not been overly precise in the above few paragraphs, but the result is that the linear equivalence in Theorem 4.18 becomes the following identity.

$$Y \cap D(m_1 m_2, p_1 p_2) \sim Y \cap D(m_1 p_1, m_2 p_2) \tag{12}$$

The strategy of the remaining part of the proof is to count the number of points, or possible maps, in each side of the linear equivalence. The desired recursive relationship (1) will be what we get when setting the number of such maps on either side of the equivalence to be equal. For motivation on why we expect each side to have the same number of points, consider the isomorphism found in Example 4.11, where the number of points uniquely determines the divisor up to linear equivalence.

Example 5.1. We will count the maps in the intersection for the $d=2$ case. Kock and Vainsencher do this in their proof [4, Proposition 3.2.2], and it is an effective way to see how the argument works.

First, let us list each of the irreducible boundary divisors that make up $D(m_1m_2, p_1p_2)$, as is done in Figure 9. We will look at each irreducible boundary divisor individually, and count the number of maps in the intersection. It is, fortunately, the case that the intersection of any two distinct irreducible boundary divisors and Y is empty, so this is not something we need to consider. This follows because such an intersection has codimension 2 in Y (see Example 4.6), and is thus empty.

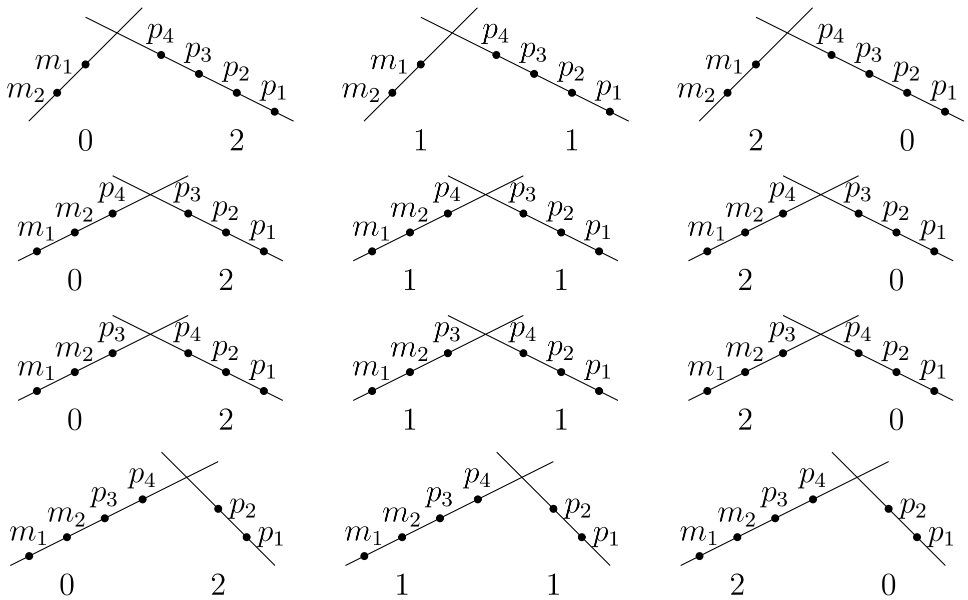

Figure 9: [4, Proposition 3.2.2] The irreducible boundary divisors that sum to $D(m_1m_2, p_1p_2)$, arranged by the distribution of the two marks p_3, p_4 and the distribution of the degree between the twigs.

The first of these is the top-left boundary divisor. The left twig is of degree-0, so the entire twig is mapped to a single point. Since m_1, m_2 lie on this twig, the image must be the unique point in the intersection $L_1 \cap L_2$. Thus from the perspective of the right twig, we are seeking a map $\mathbb{P}^1 \to \mathbb{P}^2$ which passes through the images of all the marks p_i, so the points $Q_i \in \mathbb{P}^2$, and the image of the node, which we have just found to be the intersection $L_1 \cap L_2 \in \mathbb{P}^2$. By the correspondence between parametrisations and rational curves themselves, which was informally stated in the introduction, the number of admissible maps in this intersection is precisely N_2.

The remaining divisors in the first column must have empty intersection, for the left twig is mapped to the intersection $L_1 \cap L_2$, which, by generality, will not coincide with $\mu(p_3) = Q_3$ or $\mu(p_4) = Q_4$.

Similarly, all of the divisors in the right column will have empty intersection with Y, for the right twig maps to a point, but $\mu(p_1) = Q_1$ and $\mu(p_2) = Q_2$ are distinct points by generality.

For the middle column, consider that no three of the points p_i can lie on the same twig, for otherwise the twig is mapped to a line containing three of the points Q_i, and this contradicts the generality of the points Q_i. Thus the top three divisors in the middle column will have empty intersection with Y.

For the bottom divisor in the middle column, the right twig is a degree-1 map which passes through two fixed points $\mu(p_1) = Q_1$ and $\mu(p_2) = Q_2$. We know there is at most one such map, but recall that this information is captured by the number N_1. Similarly, the left twig has two marks p_3, p_4 which are mapped to the two fixed points Q_3, Q_4. The extra marks m_j on this twig add no constraints, because any line through Q_3, Q_4 has a uniquely defined intersection with each line L_j, the preimage of which must be m_j. Thus there are $N_1 = 1$ maps from the left twig also. Much like the marks m_j on the left twig, the node also imposes no constraints, so there is precisely $N_1 \cdot N_1 = 1$ map in the intersection of this divisor and Y.

Thus we have shown that the size of the intersection is

$$|Y \cap D(m_1 m_2, p_1 p_2)| = N_2 + N_1 \cdot N_1$$

We will now find the maps in the right-hand side of (12). Our strategy is much the same as for the left-hand side as above, but we will not draw out the table.

Each of the twigs contains one of the marks p_i which maps to a point Q_i and a mark m_j which maps to one of the lines L_j. Thus it follows that neither twig will have a degree-0 map from it, for this contradicts $Q_i \notin L_j$. In other words, the divisors that will contribute to the intersection must have a degree-1 map from each twig.

As before, we must have at most two of the marks p_i on any one twig, for with three on one twig, we would contradict the generality of the points Q_i. It follows that the divisors that contribute to the intersection are those in Figure 10, and they each contribute $N_1 \cdot N_1 = 1$ point to the intersection. This is because the marks m_j and the node impose no restrictions on the maps in the intersection, and so the number of maps in the intersection is the number of lines through p_1, p_3 times the number of lines through p_2, p_4, in the case of the left divisor in Figure 10.

We now have found the number of points in the right-hand side of the equation (12) as

$$|Y \cap D(m_1 p_1, m_2 p_2)| = 2 \cdot N_1 \cdot N_1$$

In particular, we have found $N_2 + N_1 \cdot N_1 = 2 \cdot N_1 \cdot N_1$, which is precisely the $d = 2$ case of the formula (1). We conclude $N_2 = 1$ as desired.

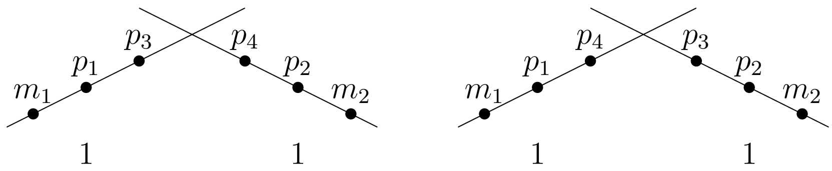

Figure 10: The irreducible boundary divisors that each contribute $N_1 \cdot N_1 = 1$ map to the intersection of Y and the boundary divisor $D(m_1p_1, m_2p_2)$.

It remains for us to generalise the $d = 2$ case in Example 5.1 for all $d \geq 2$. It is useful to see the $d = 3$ case of the argument, which sees the beginning of where the combinatorial coefficients appear in (1). The reader is advised to see Kock and Vainsencher's argument, found in [4, Proposition 3.2.3].

We will first examine the intersection $Y \cap D(m_1m_2, p_1p_2)$. Throughout, we will consider the divisor $D(A, B; d_A, d_B)$. There are three cases to consider: $d_A = 0$, which contributes N_d maps to the intersection; $d_B = 0$, which has no contribution; and $d_A, d_B \geq 1$, which contributes the sum in the left-hand side of (1).

Suppose $d_A = 0$. Then m_1, m_2 map to the same point $P \in L_1 \cap L_2$, and hence, by generality, we can assume all of the other marks lie on the B-twig. Then the A-twig doesn't really make any difference to the number of points in the intersection, for all configurations of this twig are isomorphic. By the recursive structure in Proposition 4.16, we are really counting the number of maps of degree $d_B = d$ on B with the $3d - 1$ special points, being the marks p_i and the node, mapping to prescribed points Q_i and P in $\mathbb{P}^2$. This information is described by N_d.

Suppose $d_B = 0$. Then the two marks p_1, p_2 each map to the same point of $\mathbb{P}^2$, contradicting generality. Thus no divisor with $d_B = 0$ intersects non-trivially with Y.

Suppose $d_A, d_B \geq 1$. We have to distribute $3d - 4$ marks among the two twigs. By generality, we cannot have more than $3d_A$ of the marks lying on the A-twig. Otherwise, there is a degree-d_A curve passing through $3d_A$ of the points Q_i, which is impossible by our assumption. Similarly, we have at most $3d_B - 1$ marks on the B-twig. We know that p_1, p_2 are on the B-twig, so the B-twig contains at most $3d_B - 3$ of the points we are to distribute between the twigs. Combining these inequalities, we see that this determines the number of marks on each twig, for

$$3d - 4 = (3d_A - 1) + (3d_B - 3)$$

There are $\binom{3d-4}{3d_A-1}$ ways to distribute the marks between the twigs.

Ignoring the marks m_j and the node for a moment, we see that there are N_{d_A} maps on the A-twig that pass through the points as desired, and N_{d_B} maps on the B-twig. Given any two such maps, the intersection of L_j with the image of the A-twig will have d_A points with multiplicity, by Bézout's Theorem, and we can choose m_j to be in the preimage of any of them. Thus there are d_A^2 ways to choose the positions of the two marks m_1, m_2.

Similarly, using the recursive structure from Proposition 4.16, the node must map to a point in the intersection of the images of the A and B-twigs. By Bézout's Theorem again, there are $d_A d_B$ points with multiplicity in this intersection, and this corresponds to this many choices of where to put the node. By 'choosing where to put the node,' we mean choosing where to place the mark x in Proposition 4.16.

In summary,

$$|Y \cap D(m_1 m_2, p_1 p_2)| = N_d + \sum_{\substack{d_A + d_B = d \\ d_A, d_B \geq 1}} \binom{3d-4}{3d_A - 1} \cdot d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_B \tag{13}$$

Next we have to find the number of maps in $Y \cap D(m_1 p_1, m_2 p_2)$. Again, we will consider the divisor $D(A, B; d_A, d_B)$. The different cases of d_A, d_B make the following contributions to the intersection: if $d_A = 0$ or $d_B = 0$, then there is no contribution; and if $d_A, d_B \geq 1$, we get the sum on the right-hand side of (1).

Each twig contains a mark m_j and a mark p_i, so if one of d_A or d_B is zero, these marks must map to the same point in $\mathbb{P}^2$. But this contradicts generality, for $Q_i = \mu(p_i) \notin L_j$ but $\mu(m_j) \in L_j$ This shows that when d_A or d_B is zero, we get no contribution to the intersection.

Suppose that $d_A, d_B \geq 1$. As before, we have $3d - 4$ marks to distribute, and no $3d_A$ (respectively $3d_B$) of the marks p_i can lie on the A-twig (respectively B-twig). This means that, of the $3d - 4$ marks we are distributing, we must put $3d_A - 2$ on the A-twig and $3d_B - 2$ on the B-twig. There are $\binom{3d-4}{3d_A-2}$ such distributions of the marks.

Again, ignore the node and the marks m_j for a moment. On the A-twig, we have $3d_A - 1$ marks p_i, so there are N_{d_A} maps here. When choosing where to place m_1 on the A-twig, we can place m_1 anywhere in the preimage of the intersection of L_1 and the image of A. By Bézout's Theorem, there are d_A points in this intersection, counting multiplicity. Similarly, there are $d_B N_{d_B}$ maps on the B-twig with a choice of the mark m_2.

Finally, the node is placed in the preimage of the intersection of the images of the A and B-twigs, as before. By Bézout's Theorem, there are $d_A d_B$ points to choose from, counting multiplicity.

We have found, therefore, that

$$|Y \cap D(m_1p_1, m_2p_2)| = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-2} \cdot d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B \tag{14}$$

Equating (13) and (14) gives us (1), completing the proof. □

6 Conclusion

In this dissertation, we have introduced the notions of fine and coarse moduli spaces. We have looked at two examples of moduli spaces, those of stable curves and of stable maps, and have used their structures to find a recursive formula for the number of rational curves passing through general points in the projective plane. Our proof involved counting the intersection of a number of preimages and boundary divisors.

6.1 Enumerative Geometry

This question and method of proof is common in the field of enumerative geometry, which, in Kock and Vainsencher's words, "[aims to] *count how many geometric figures satisfy given conditions*" [4, Introduction]. The strategy of proof is as follows: create a natural correspondence between the objects you are considering and an algebraic variety M, such that conditions on the objects cut out subvarieties of M. The answer to the geometric question is then answered by looking at and examining the intersection of the subvarieties, so that we now have a problem in intersection theory.

In our case, we were looking at stable $(3d-1)$-pointed maps from a tree of projective lines to $\mathbb{P}^2$ of degree d, and these are in natural correspondence, by Definition 3.9, with the coarse moduli space $\overline{M}_{0,3d-1}(\mathbb{P}^2, d)$. We were asking the question of how many of these maps pass through fixed general points $P_1, \ldots, P_{3d-1} \in \mathbb{P}^2$. To impose that the i-th mark is mapped to P_i is to cut out a subvariety $\nu_i^{-1}(P_i) \subset \overline{M}_{0,3d-1}(\mathbb{P}^2, d)$. Finally, the number of maps through the points P_i is precisely the number of points in the intersection of the subvarieties $\nu_i^{-1}(P_i)$. Thus our geometric question is converted into a problem in intersection theory.

Of course, in order to compute the number of points in the intersection, we actually looked at a different intersection in a larger moduli space $\overline{M}_{0,3d}(\mathbb{P}^2, d)$, but this can be considered to be the method of solving the rephrased problem. Indeed, we have an isomorphism

$$\begin{aligned} D(m_1 m_2, p_1 \ldots p_{3d-2}; 0, d) &\cong \overline{M}_{0,3d-1}(\mathbb{P}^2, d) \times_{\mathbb{P}^2} \mathbb{P}^2 \\ &\cong \overline{M}_{0,3d-1}(\mathbb{P}^2, d) \end{aligned}$$

from Proposition 4.16, using that $\overline{M}_{0,3}(\mathbb{P}^2, 0) \cong \overline{M}_{0,3} \times \mathbb{P}^2 \cong \mathbb{P}^2$. This isomorphism naturally identifies the moduli space constructed in the paragraph above with the structures we used in our proof.

6.2 Gromov-Witten Invariants and Quantum Cohomology

We aim to sketch an introduction to Quantum Cohomology. We will largely follow the exposition of Kock and Vainsencher in [4, Sections 3-5].

In our discussion of divisors in Section 4.1, we restricted our discussion to codimension 1 subvarieties, but we can consider subvarieties of all codimensions with coefficients in $\mathbb{Q}$, and form the *Chow group* $A_*(\mathbb{P}^r) = \bigoplus_{k=0}^r A_k(\mathbb{P}^r)$. In our notation from Section 4.1, $A_{r-1}(\mathbb{P}^r) = \mathrm{Cl}\,(\mathbb{P}^r) \otimes \mathbb{Q}$. The intersection ring $A^*(\mathbb{P}^r)$ is defined by setting $A^k(\mathbb{P}^r) = A_{r-k}(\mathbb{P}^r)$ via a Poincaré duality isomorphism

$$\begin{aligned} A^*(\mathbb{P}^r) &\to A_*(\mathbb{P}^r) \\ \gamma &\mapsto \gamma \cap [\mathbb{P}^r] \end{aligned}$$

and using intersections to define multiplication, so that $A^*(\mathbb{P}^r)$ has the structure of a graded $\mathbb{Q}$-algebra. For $\mathbb{P}^r$, there is a natural isomorphism

$$A^*(\mathbb{P}^r) \cong \frac{\mathbb{Q}[h]}{(h^{r+1})}$$

and we can use $\{h^0, \ldots, h^r\}$ as a basis for $A^*(\mathbb{P}^r)$, where $h^0 \in A^0(\mathbb{P}^r) = A_r(\mathbb{P}^r)$ corresponds to the whole space $\mathbb{P}^r$, and $h^r \in A^r(\mathbb{P}^r) = A_0(\mathbb{P}^r)$ corresponds to a point.

The intersection of subvarieties $\cap_i \nu_i^{-1}(P_i)$ we considered in Section 6.1 can immediately be generalised to the intersection $\cap_i \nu_i^{-1}(\Gamma_i)$, where $\Gamma_i \subset \mathbb{P}^r$ are irreducible subvarieties. If the subvarieties $\Gamma_i \subset \mathbb{P}^r$ are sufficiently general and

$$\sum_i \mathrm{codim}(\Gamma_i) = \dim \overline{M}_{0,n}(\mathbb{P}^r, d) \tag{15}$$

the intersection consists of a finite number of points [4, Proposition 3.4.3]. Let $\gamma_i \in A^*(\mathbb{P}^r)$ correspond to $\Gamma_i \in A_*(\mathbb{P}^r)$. We define the *Gromov-Witten invariants* $I_d(\gamma_1 \cdots \gamma_n)$ to be the number of rational curves of degree d which go through all of the subvarieties Γ_i. This definition only works with the above assumptions and only if $\mathrm{codim}(\Gamma_i) \geq 2$, but the Gromov-Witten invariants can be defined without these assumptions (the full definition and proof that it corresponds to the above can be found in [4, Proposition 4.1.5]).

For example, let us consider $\overline{M}_{0,n}(\mathbb{P}^2, d)$, which has dimension $3d - 1 + n$ by Proposition 3.13. If $n = 3d-1$, then the general points $P_1, \ldots, P_{3d-1}$, each of codimension 2, will satisfy (15). The points correspond to $h^2 \in A^2(\mathbb{P}^2)$, so

$$N_d = I_d(\underbrace{h^2 \cdots h^2}_{3d-1 \text{ times}})$$

Multiplication in the Chow ring $A^*(\mathbb{P}^r)$ corresponds to intersections, and can actually be written in terms of some of the Gromov-Witten invariants

$$h^i \cup h^j = \sum_{e+f=r} I_0(h^i \cdot h^j \cdot h^e)\, h^f \in A^{i+j}(\mathbb{P}^r) \tag{16}$$

Sometimes, this *cup product* $\cup$ is too limited: for example, if $i + j > r$, the product is zero. The *quantum cohomology ring* has the following *quantum product*

$$h^i * h^j = \sum_{e+f=r} \Phi_{ije}\, h^f$$

where

$$\Phi_{ije} = \sum_{\substack{a_0,\dots,a_r \in \mathbb{N} \\ d \in \mathbb{N}}} \frac{x_0^{a_0} \cdots x_r^{a_r}}{a_0! \cdots a_r!}\, I_d \left(\underbrace{(h^0 \cdots h^0)}_{a_0 \text{ times}} \cdots \underbrace{(h^r \cdots h^r)}_{a_r \text{ times}} \cdot h^i \cdot h^j \cdot h^e \right)$$

It is a fact that this sum has only finitely many nonzero terms. Notice how we have introduced formal variables $x_0, \dots, x_r$, so we are now working in a $\mathbb{Q}[[x_0, \dots, x_r]]$-algebra. When $a_0, \dots, a_r = 0, d = 0$, we get the term corresponding to the classical product (16).

We will finish by saying that the quantum product is both commutative and associative, and one can extract results such as Kontsevich's formula from the associativity relations.

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