Introduction to Higher Categories

Todd Liebenschutz-Jones

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Abstract

This is a short introduction to higher categories based on [Bae97] for submission as a broadening project for the C2.7 Category Theory course.

1 Introduction

This short introduction to higher categories uses [Bae97] as its principal reference. In Section 2, we introduce the idea of morphisms between morphisms and the notion of a strict 2-category from [Bae97, §2], using the examples from [Bae97, §3.1]. In Section 3, we examine the example of [Bae97, §3.2] to introduce weak n-categories, which consider morphisms only up to isomorphisms between them.

As [Bae97] explains, providing a general definition of an n-category is challenging, and in these notes, we do not examine some of the possible choices to take as definitions. In particular, we do not discuss choosing different shapes for our n-morphisms, and for us, all n-morphisms occur on a bigon[1] between two $(n-1)$-morphisms.

2 The category of categories

One of the simplest and most natural examples of a category is the category[2] Set of sets. Its objects are the sets and its morphisms are the maps between such sets. Using the maps, we are able to define the notion of an isomorphism of two sets, and this allows us to identify two sets which are *essentially the same*, but which technically are not equal as sets. This is an incredibly powerful notion, and is one of the key ideas behind definitions using universal properties, in which we only define objects *up to isomorphism*.

Another natural example is the category Cat of categories, whose objects are categories and whose morphisms are functors. However, this category carries additional maps, namely the natural transformations between functors. These natural transformations admit two different kinds of composition, namely horizontal and vertical composition (see for example [SK18, §1.6]), and there are also identity natural transformations. This additional structure of natural transformations between functors makes Cat into a strict 2-category, and the natural transformations are 2-morphisms.

[1] A *bigon* is a shape with two verticies and two edges between them.
[2] Throughout this report, we will suppress the notation of *small* set, etc.

Definition 2.1. A *strict 2-category* $\mathcal{C}$ is a usual category $\mathcal{C}$ together with so-called *2-morphisms*, which are arrows between morphisms of $\mathcal{C}$. These 2-morphisms are denoted by double arrows as per the standard notation for natural transformations. Similar to the definition of a category, these 2-morphisms have associative horizontal and vertical composition operations, and identity 2-morphisms for every morphism of $\mathcal{C}$. In addition, the different composition operations are related using the *interchange law*

$$(a \cdot b)(c \cdot d) = (ac) \cdot (bd) \tag{2.1}$$

where

$$\begin{tikzcd} X \arrow[r, bend left=60, ""{name=U1}] \arrow[r, ""{name=M1}] \arrow[r, bend right=60, ""{name=D1}] & Y \arrow[r, bend left=60, ""{name=U2}] \arrow[r, ""{name=M2}] \arrow[r, bend right=60, ""{name=D2}] & Z \arrow[Rightarrow, from=U1, to=M1, phantom, "\Downarrow a"] \arrow[Rightarrow, from=M1, to=D1, phantom, "\Downarrow b"] \arrow[Rightarrow, from=U2, to=M2, phantom, "\Downarrow c"] \arrow[Rightarrow, from=M2, to=D2, phantom, "\Downarrow d"] \end{tikzcd} \tag{2.2}$$

Definition 2.1 may be extended indefinitely, yielding a *strict n-category* with 3-morphisms between its 2-morphisms, 4-morphisms between its 3-morphisms all the way up to n-morphisms between its $(n-1)$-morphisms. All the k-morphisms have various different associative composition operations which appropriately intertwine and there are identity k-morphisms for every $(k-1)$-morphism.

3 Equality of morphisms

Consider the category Top of topological spaces together with continuous maps. This category also carries a natural notion of a 2-morphism, namely a homotopy between continuous maps. We can continue this idea recursively to find a natural notion of n-morphism on Top as a homotopy between $(n-1)$-morphisms. Thus an n-morphism is a continuous map $[0,1]^{n-1} \times X \to Y$, where $X, Y \in \text{Top}$ are topological spaces and $[0,1]^{n-1}$ is the $(n-1)$-dimensional hypercube.

One might expect these natural k-morphisms to form a strict n-category, however this is not the case for the following reason. Consider continuous maps $f_i : X \to Y$ for $i = 0, \ldots, 3$ and homotopies $H_i : [0,1] \times X \to Y$ from $f_i = H_i|_0$ to $f_{i+1} = H_i|_1$. We draw this in the following diagram

$$\begin{tikzcd} X \arrow[r, bend left=80, "f_0", ""{name=A}] \arrow[r, bend left=20, "f_1"', ""{name=B}] \arrow[r, bend right=20, "f_2", ""{name=C}] \arrow[r, bend right=80, "f_3"', ""{name=D}] & Y \arrow[from=A, to=B, phantom, "\Downarrow H_0"] \arrow[from=B, to=C, phantom, "\Downarrow H_1"] \arrow[from=C, to=D, phantom, "\Downarrow H_2"] \end{tikzcd} \tag{3.1}$$

We can compose two homotopies by, for example, defining

$$\begin{aligned} (H_1 \circ H_0) : [0,1] \times X &\to Y \\ (t, x) &\mapsto \begin{cases} H_0(2t, x) & t \leq 1/2 \\ H_1(2t-1, x) & t \geq 1/2 \end{cases} \end{aligned} \tag{3.2}$$

This definition does not yield an associative composition operation however, since in general

$$H_2 \circ (H_1 \circ H_0) \neq (H_2 \circ H_1) \circ H_0 \tag{3.3}$$

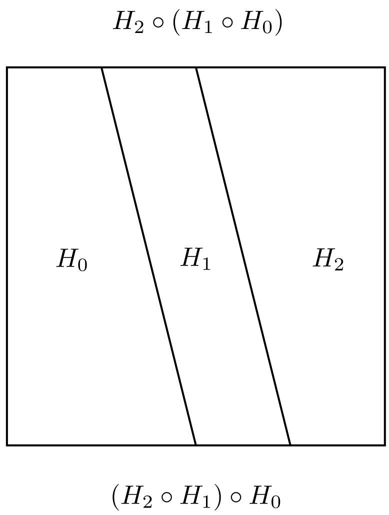

Figure 1: There is a homotopy between the two sides of (3.3).

Instead, this composition operation is associative *up to homotopy*, and the standard homotopy here is represented schematically in Figure 1.

This can be seen as motivation for the definition of *weak n-categories*. In a weak 2-category, we no longer require that morphisms (i.e. 1-morphisms) are associative *on the nose*, and instead require only that they are associative up to invertible 2-morphism. Similarly, composition with the identity 1-morphisms only needs to be the identity map up to invertible 2-morphism. More generally, in a weak n-category, we require associativity and identity composition of k-morphisms to hold only up to invertible $(k+1)$-morphism for all $k < n$. Capturing the new requirements for weak n-categories is a pain, and we happily do not include the details here.

Let us consider what this means for Top. As discussed in Section 2, we use isomorphisms to let us identify objects when they are, to all intents and purposes, the same. For Top (as a 1-category), this means we look at topological spaces *up to homeomorphism*. When we look at Top as a ω-category[3] we see that two morphisms $f : X \to Y$ and $g : Y \to X$ are isomorphisms if and only if $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. This now means we look at topological spaces *up to homotopy equivalence*.

References

[Bae97] John C. Baez. An introduction to n-categories. In *Category theory and computer science (Santa Margherita Ligure, 1997)*, volume 1290 of *Lecture Notes in Comput. Sci.*, pages 1–33. Springer, Berlin, 1997.

[SK18] Pavel Safronov and Frances Kirwan. C2.7: Category Theory lecture notes, 2018.

[3] So we have n-morphisms for all $n \in \omega$, which all satisfy associativity and identity composition up to invertible $(n+1)$-morphisms.