The Atiyah-Singer Index Theorem

Todd Liebenschutz-Jones

January 4, 2018

Contents

1 Introduction

The Atiyah-Singer Index Theorem offers a method to calculate the Fredholm index of a differential operator on a manifold under certain hypotheses. The theorem asserts the equality of the Fredholm index, referred sometimes to as the *analytical index*, and the *topological index* associated to the differential operator. The topological index depends only on the topology of the manifold and the *symbol* of the operator, and is readily computable. By contrast, the analytical index is hard to compute directly.

In this project, we introduce the reader to differential operators without assuming prior knowledge of this topic. We discuss why the differential operators are Fredholm, under the appropriate hypotheses, and in Section 5, we explain why the Fredholm index should only depend on the symbol of the differential operator, and the underlying topology of the manifold.

1.1 Prerequisites

This project assumes understanding of smooth manifolds, including familiarity with partition of unity arguments. A first course in Functional Analysis, including Fredholm operators, is also required.

2 Differential Operators

On a subset $U \subset \mathbb{R}^n$, a differential operator $P : C^\infty(U) \to C^\infty(U)$ is a finite sum

$$Pu = \sum_{|\alpha| \leq r} a_\alpha D^\alpha u \tag{1}$$

where for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have a function $a_\alpha \in C^\infty(U)$ and an operator

$$D^\alpha = (D_1)^{\alpha_1} \cdots (D_n)^{\alpha_n}$$

Here, we use the notation of [1],

$$D_k = -i \left(\frac{\partial}{\partial x_k} \right)$$

It is easy to check the nontrivial fact that an operator is a differential operator if, and only if, it is a differential operator with respect to any local coordinates. Thus the definition above extends naturally to the following notion. A differential operator on a manifold X is a map $C^\infty(X) \to C^\infty(X)$ which has the form (1) on every coordinate patch.

2.1 Symbols and Ellipticity

The symbol of a differential operator (1) characterises the highest order derivatives. Precisely, the symbol is $\sigma(P) : U \times \mathbb{R}^n \to \mathbb{C}$ given by

$$(x, \xi) \mapsto \sum_{|\alpha|=r} a_\alpha(x)\xi^\alpha \tag{2}$$

This notion naturally extends to the symbol of a differential operator on a manifold, in which the symbol is a map $\sigma : T^*(X) \to \mathbb{C}$ from the cotangent space, with the local form (2) on each coordinate patch.

The differential operator is elliptic when the symbol is nonzero at all (x, ξ) where $\xi \neq 0$. This corresponds to the symbol being nonzero away from the zero section of the cotangent bundle.

2.2 Inner Product

Suppose that $\mathrm{d}x$ is a non-vanishing smooth measure on X, meaning that on each coordinate patch $(U, x_1, \ldots, x_n)$, the restriction of the measure is

$$\mathrm{d}x = \varphi \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

where φ is a positive C^∞ function. This measure induces an integral on the manifold X which we will use to construct an inner product. For a compact orientable manifold X, fixing a nonvanishing n-form of volume 1 would suffice, however we can prove the following theorem without these stronger assumptions.

Define the L^2 inner product on $C^\infty(X)$ by $\langle u, v \rangle = \int u\overline{v}\,\mathrm{d}x$, for all $u, v \in C^\infty(X)$.

Theorem 2.1. [1, Theorem 2.4] *Given a differential operator P of order m, there is a unique differential operator P^t of the same order with the property*

$$\langle Pu, v \rangle = \langle u, P^t v \rangle$$

for all $u, v \in C^\infty(X)$.

Proof. We follow the proof of [1]. On a coordinate patch $(U, x_1, \ldots, x_n)$, we write $P = \sum a_\alpha D^\alpha$ and $\mathrm{d}x = \varphi \mathrm{d}x_1 \cdots \mathrm{d}x_n$. Observe

$$0 = \int \frac{\partial(u\overline{v})}{\partial x_k} \mathrm{d}x = \left\langle \frac{\partial u}{\partial x_k}, v \right\rangle_U + \left\langle u, \frac{\partial v}{\partial x_k} \right\rangle_U$$

so that $\langle D^\alpha u, v\rangle_U = \langle u, D^\alpha v\rangle_U$. Thus

$$\begin{aligned}
\langle Pu, v\rangle_U &= \sum_\alpha \int a_\alpha D^\alpha(u)\overline{v}\varphi \mathrm{d}x_1 \ldots x_n \\
&= \sum_\alpha \int D^\alpha(u)\overline{\overline{a}_\alpha \varphi v} \mathrm{d}x_1 \ldots x_n \\
&= \sum_\alpha \int u\overline{D^\alpha(\overline{a}_\alpha \varphi v)} \mathrm{d}x_1 \ldots x_n \\
&= \sum_\alpha \int u\overline{\frac{1}{\varphi} D^\alpha(\overline{a}_\alpha \varphi v)}\varphi \mathrm{d}x_1 \ldots x_n \\
&= \langle u, P^t v\rangle_U
\end{aligned}$$

where

$$P^t = \frac{1}{\varphi} D^\alpha(\overline{a}_\alpha \varphi v)$$

This proves the local existence and local uniqueness of P^t. A standard partition of unity argument completes the proof. □

Notice that P^t is elliptic if, and only if, P is elliptic.

3 Smoothing Operators

Definition 3.1. Let X be a compact manifold with a smooth non-vanishing measure $\mathrm{d}x$. The smoothing operator $T_K : C^\infty(X) \to C^\infty(X)$ corresponding to the function $K \in C^\infty(X \times X)$ is given by

$$(T_K f)(x) = \int K(x, y) f(y) \mathrm{d}y$$

The adjoint of T_K is the smoothing operator T_L, where $L(x, y) = \overline{K(y, x)}$, as can be seen below.

$$\begin{aligned}
\langle T_K f, g\rangle &= \int \left(\int K(x, y) f(y) \mathrm{d}y\right) \overline{g(x)} \mathrm{d}x \\
&= \int f(y) \left(\overline{\int \overline{K(x, y)} g(x) \mathrm{d}x}\right) \mathrm{d}y \\
&= \langle f, T_L g\rangle
\end{aligned}$$

4 Fredholm Theory of Differential Operators

We will show in this section that an elliptic differential operator is a Fredholm operator. Such operators are defined by their finite dimensional kernels and cokernels and their closed range. We use the inner product and adjoint P^t from Section 2.2 to phrase this property as follows.

Theorem 4.1. *Let X be a compact manifold and let P be an elliptic differential operator. Then P has finite dimensional kernel and* $\operatorname{Range} P = \ker P^{t\perp}$.

Since P^t is elliptic, it also has finite dimensional kernel, hence P has finite dimensional cokernel. Furthermore, P has closed range for the orthogonal complement of a subspace is always closed. Thus Theorem 4.1 implies P is Fredholm as desired.

The strategy of proof is to show that P is almost invertible, up to an error term, where the error term is a smoothing operator. We will appeal to the following two lemmas, which will not be proved here. See [1] for the proofs.

Lemma 4.2. [1, Theorem 3.1] *There is an operator Q and a smoothing operator T_K such that*

$$PQ = I - T_K$$

This may be quoted as P is right invertible, modulo smoothing operators.

Lemma 4.3. [1, Theorem 3.2] *The conclusions of Theorem 4.1 hold for the operator $I - T_K$. That is, $I - T_K$ has finite dimensional kernel and its range is the orthogonal complement of* $\ker(I - T_L)$*, where T_L is the adjoint smoothing operator to T_K.*

We are now ready to complete the proof of Theorem 4.1. We follow the argument provided in [1].

Proof. Let $V = \ker(I - T_K)$ be the finite dimensional subspace with orthonormal basis $f_1, \dots, f_m$. For any $f \in C^\infty(X)$, we can write $f = g + h$, where

$$g = \sum \langle f, f_i \rangle f_i \in V$$

and $h = f - g \in V^\perp$.

Since $\operatorname{Range} P \subseteq \operatorname{Range}(I - T_K)$ by Lemma 4.2, define $U = (V \cap \operatorname{Range} P)^\perp \cap V$. We want to show that $C^\infty(X)$ is the orthogonal direct sum of U and $\operatorname{Range} P$. With $f = g + h$ as above, consider that $h \perp V$ implies $h \in \operatorname{Range}(I - T_K)$ by Lemma 4.3, and hence $h \in \operatorname{Range} P$. On the other hand, $g \in V$ splits as $f_1 + g_2$, where $f_1 \in U$ and $g_2 \perp U$, so that $g_2 \in \operatorname{Range} P \cap V$ by definition of U. Write $f_2 = g_2 + h$ to conclude.

Thus U is a finite dimensional space whose orthogonal complement in $C^\infty(X)$ is $\operatorname{Range} P$. It remains to show that $U = \ker P^t$:

$$\begin{aligned} f \in U &\iff f \perp \operatorname{Range} P \\ &\iff \langle f, Pu \rangle = 0 \text{ for all } u \in C^\infty(X) \\ &\iff \langle P^t f, u \rangle = 0 \text{ for all } u \in C^\infty(X) \\ &\iff P^t f = 0 \end{aligned}$$

By switching P and P^t, we immediately see that P has a finite dimensional kernel.

□

5 Atiyah-Singer Index Theorem

Let X be a compact manifold. Consider two elliptic differential operators P and Q on X with the same symbol σ. Using a standard partition of unity argument, we can consider a homotopy of elliptic differential operators P_t for $t \in [0, 1]$ such that $P_0 = P$, $P_1 = Q$ and moreover that each P_t is an elliptic differential operator with the same symbol σ. Recall from [3, Theorem 11.6] that the index map is a locally constant map from the space of Fredholm operators. It follows that the operators P and Q have the same Fredholm index, and more generally that the index of an elliptic differential operator depends only on its symbol.

The Atiyah-Singer Index Theorem offers a method to calculate the index of such an operator P using the topological properties of the manifold X and the symbol of P. Its statement under certain hypotheses is given below, but it applies more generally.

Theorem 5.1. [2, Theorem 5.1] *Let P be a linear elliptic partial differential operator on a smooth, closed even-dimensional manifold X. Let σ be the symbol of P. Associate to σ its symbol class $[\sigma] \in K(T^*X)$ from K-theory. The Fredholm index of P is given by*

$$\operatorname{Index}(P) = \int_{T^*X} \operatorname{ch}[\sigma] \operatorname{Todd}(TX \otimes \mathbb{C}) \tag{3}$$

where ch *denotes the Chern character, and* $\operatorname{Todd}(TX \otimes \mathbb{C})$ *is the Todd class of the vector bundle* $TX \otimes \mathbb{C}$.

We will not define the ingredients in the right hand side of (3) in this project, but while mathematically complex, the right hand side is in fact readily computable.

References

[1] Victor Guillemin. Elliptic operators, 2005.

[2] Nigel Higson and John Roe. Lectures on operator k-theory and the atiyah-singer index theorem, 2004.

[3] David Seifert. C4.1 functional analysis, 2017.