Equivariant Seidel maps and a flat connection on equivariant symplectic cohomology

UNIVERSITY OF OXFORD
Mathematical Institute

Todd Liebenschutz-Jones
DPhil student
Mathematical Institute, University of Oxford

Geometry and Analysis seminar, 26 April 2021

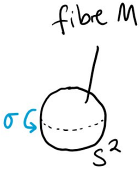
An intertwining relation for equivariant Seidel maps
arXiv:2010.03342

Shift operators and connections on equivariant symplectic cohomology
arXiv:2104.01891

Oxford
Mathematics

(M, ω) is a closed symplectic manifold.
$\sigma : S^1 \times M \to M$ is a Hamiltonian S^1-action.

The **clutching bundle** $E(\sigma)$ is a bundle with fibre M over the sphere S^2.



The **quantum Seidel map**

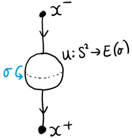
$$Q\mathcal{S}(\sigma) : QH^*(M) \to QH^{*+|\sigma|}(M)$$

counts pseudoholomorphic sections of the clutching bundle $E(\sigma) \to S^2$.

Novikov ring is $\Lambda = \mathbb{Z}[q^{H_2(M)}] \ni q^A$ with $A \in H_2(M)$.
Quantum cohomology is $QH^*(M) = H^*(M) \otimes \Lambda$.

Novikov ring is $\Lambda = \mathbb{Z}[q^{H_2(M)}] \ni q^A$ with $A \in H_2(M)$.
Quantum cohomology is $QH^*(M) = H^*(M) \otimes \Lambda$.

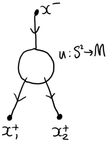
$$QS(\sigma)(x^+) = \sum_{\substack{A \in H_2(M) \\ x^- \in H^*(M)}} \#\Bigg($$



$$\Bigg) q^A x^-$$

Quantum product is

$$x_1^+ * x_2^+ = \sum_{\substack{A \in H_2(M) \\ x^- \in H^*(M)}} \#\Bigg($$



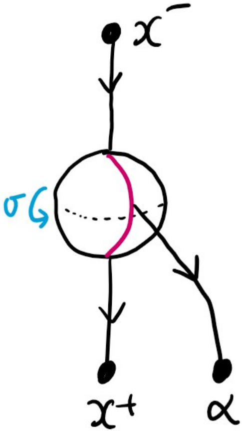
$$\Bigg) q^A x^-$$

Theorem (Seidel, '97)

We have $QS(\sigma)(\alpha * x^+) = \alpha * QS(\sigma)(x^+)$ *for* $\alpha \in QH^*(M)$.

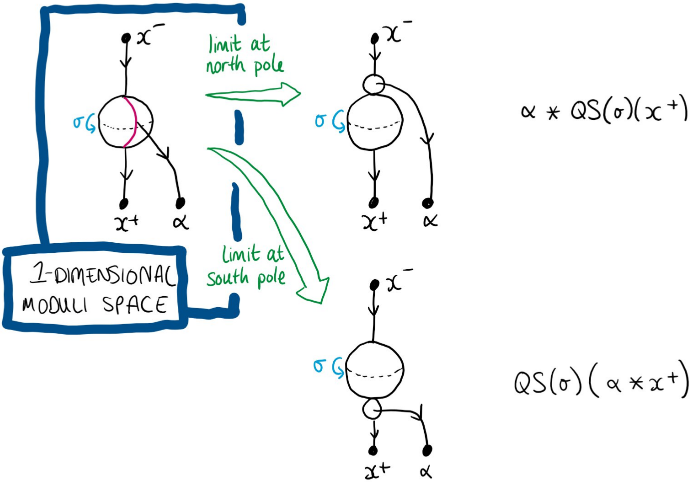
Seidel maps

Proof of $Q\mathcal{S}(\sigma)(\alpha * x^+) = \alpha * Q\mathcal{S}(\sigma)(x^+)$



Seidel maps

Proof of $Q\mathcal{S}(\sigma)(\alpha * x^+) = \alpha * Q\mathcal{S}(\sigma)(x^+)$

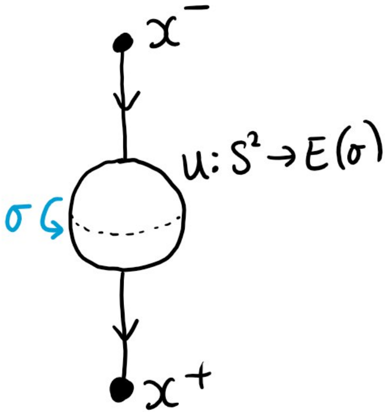


We showed $Q\mathcal{S}(\sigma)(\alpha * x^+) = \alpha * Q\mathcal{S}(\sigma)(x^+)$.
Therefore, we have $Q\mathcal{S}(\sigma)(x) = x * Q\mathcal{S}(\sigma)(1)$.
This yields the **Seidel representation**

$$\begin{aligned} \Big\{\text{Hamiltonian } S^1\text{-actions}\Big\} &\to QH^*(M)^\times \\ \sigma &\mapsto Q\mathcal{S}(\sigma)(1). \end{aligned} \tag{1}$$

Equivariant Seidel maps

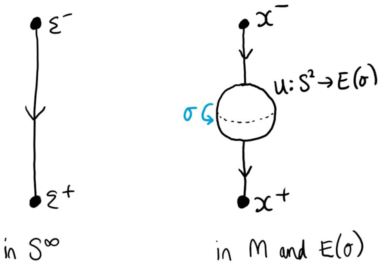
Equivariant cohomology



Equivariant Seidel maps
Definition

In the Borel construction, we have a fibre bundle

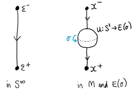
$$X \longrightarrow S^\infty \times_{S^1} X \longrightarrow \frac{S^\infty}{S^1} = \mathbb{CP}^\infty$$



S^1-**equivariant quantum Seidel map** is

$$Q\mathcal{S}_{S^1}(\sigma) : QH^*_{S^1}(M,\sigma) \to QH^{*+|\sigma|}_{S^1}(M,\mathrm{Id})$$

$$Q\mathcal{S}_{S^1}(\sigma)(\varepsilon^+, x^+) = \sum_{\substack{A \in H_2(M) \\ (\varepsilon^-, x^-) \in H^*_{S^1}(M,\mathrm{Id})}} \#\Big(\quad \Big) q^A (\varepsilon^-, x^-)$$



Theorem (Intertwining relation, TL-J)

We have

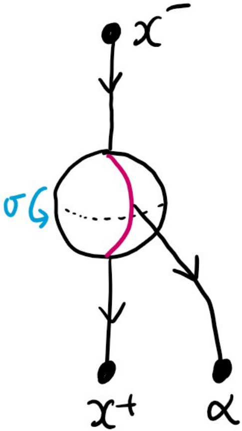
$$Q\mathcal{S}_{S^1}(\sigma)(x * \alpha^+) - Q\mathcal{S}_{S^1}(\sigma)(x) * \alpha^- = \mathbf{u}\ WQ\mathcal{S}_{S^1}(\sigma,\alpha)(x) \quad (2)$$

for any class $\alpha \in H^2_{S^1}(E(\sigma))$.
Here, $\alpha^\pm$ are the restrictions of α to the fibres above the poles, $\mathbf{u} \in H^2(\mathbb{CP}^\infty)$ is the generator of $H^*(\mathbb{CP}^\infty)$, and $WQ\mathcal{S}_{S^1}$ is a weighted version of the S^1-equivariant quantum Seidel map.

We proved this for all $\alpha \in H^*_{S^1}(E(\sigma))$ but we'll only present $|\alpha| = 2$ for simplicity.

Equivariant Seidel maps

Proof of $Q\mathcal{S}_{S^1}(\sigma)(x * \alpha^+) - Q\mathcal{S}_{S^1}(\sigma)(x) * \alpha^- = \mathbf{u}\ WQ\mathcal{S}_{S^1}(\sigma, \alpha)(x)$



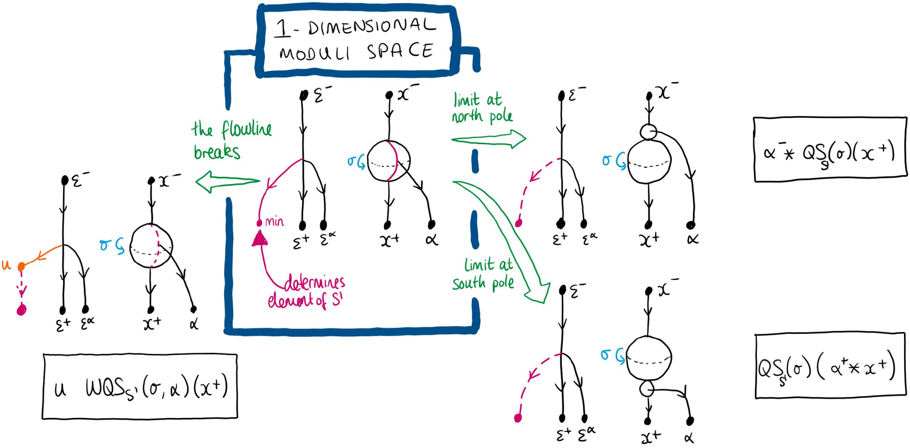
To parameterise the line of longitude, we would need an S^1-equivariant map $S^\infty \to S^1$.

But none exists.
Let $w \in S^\infty$. The composition $S^1 \cdot w \hookrightarrow S^\infty \to S^1$ is an isomorphism.
But S^∞ is contractible, so $\pi_1(S^\infty) = 0$.

Key insight: it is sufficient to define $S^\infty \to S^1$ on a *generic* subset $W \subset S^\infty$.

Equivariant Seidel maps

Proof of $\mathcal{QS}_{S^1}(\sigma)(x * \alpha^+) - \mathcal{QS}_{S^1}(\sigma)(x) * \alpha^- = \mathbf{u}\, W\mathcal{QS}_{S^1}(\sigma, \alpha)(x)$



Maulik and Okounkov defined equivariant Seidel maps in 2013.
They also proved the intertwining relation for $\alpha \in H^2_{S^1}(E(\sigma))$.
Iritani gave a similar construction in a different setting in 2017.

Their definitions use *virtual fundamental classes* to count sections.
Their proofs of the intertwining relation use *virtual localisation*.

We are interested in S^1-equivariant Floer theory, which does not have the above machinery.
We redefined $Q\mathcal{S}_{S^1}(\sigma)$ using a Morse Borel construction.
We reproved the intertwining relation with a new Morse homotopy proof using a 1-dimensional moduli space argument.

Floer cohomology

Floer Seidel map, convex manifolds, symplectic cohomology

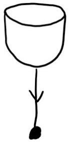
The **Floer Seidel map**

$$\mathcal{FS}(\sigma) : FH^*(M; H) \to FH^{*+|\sigma|}(M; \sigma^* H)$$

maps the Hamiltonian orbit $x : S^1 \to M$ to the orbit

$$(\sigma^* x)(t) = \sigma_t^{-1}(x(t)), \qquad t \in S^1.$$

$\mathcal{FS}(\sigma)$ is an isomorphism of cochain complexes.



Compact: $QH^*(M) \cong FH^*(M; H)$ for all Hamiltonians H.
The Floer Seidel map and the quantum Seidel map agree.

Non-compact: (with convexity assumption, for example $\mathcal{O}_{\mathbb{P}^1}(-1)$)
$FH^*(M, \lambda; H)$ depends on the slope λ ($H = \lambda R +$ const. at infinity)
Symplectic cohomology is $SH^*(M) = \varinjlim FH^*(M, \lambda)$ as $\lambda \to \infty$.
For *linear* σ of slope κ, the Floer Seidel map is

$$\mathcal{FS}(\sigma) : FH^*(M, \lambda; H) \to FH^{*+|\sigma|}(M, \lambda - \kappa; \sigma^* H).$$

The quantum Seidel map is only defined for linear σ with $\kappa \geq 0$.

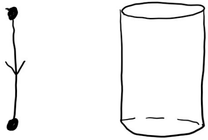
The loop space $\mathcal{L}M = \{x : S^1 \to M\}$ has an S^1-action given by

$$(\theta \cdot x)(t) = x(t - \theta) \qquad \theta \in S^1. \tag{3}$$

Definition (Equivariant Floer cohomology)

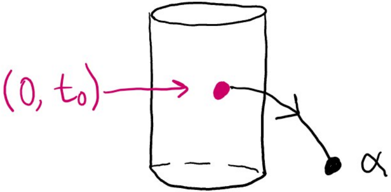
We combine Morse theory on S^∞ with Floer theory on M.
The Hamiltonian $H : S^\infty \times S^1 \times M \to \mathbb{R}$ now depends on S^∞.
The **equivariant Floer cochain complex** is generated over Λ by $[\varepsilon, x]$, where $\varepsilon \in S^\infty$ is critical and x is a Hamiltonian orbit of H_ε.

The differential counts flowlines in S^∞ paired with Floer cylinders in M.



$FH^*_{S^1}(M, \lambda; H)$ is a $\Lambda \otimes \mathbb{Z}[\mathbf{u}]$-module.

There is a map $QH^*(M) \otimes FH^*(M,\lambda) \to FH^*(M,\lambda)$ which counts

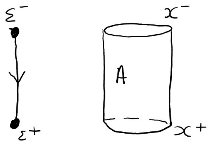


An equivariant version of this map would use a map $S^\infty \to S^1$ defined on a *generic* subset $W \subset S^\infty$.
Therefore it would not be a chain map.

Novikov ring is $\Lambda = \mathbb{Z}[q^{H_2(M)}] \ni q^A$ with $A \in H_2(M)$.
For $\alpha \in H^2(M)$, define

$$\frac{d}{d\alpha}(q^A) = \alpha(A)\, q^A. \tag{4}$$

We can pick a Λ-basis for the equivariant Floer cochain complex and apply $\frac{d}{d\alpha}$. *This is not a chain map either.*



$$\frac{d}{d\alpha}[\varepsilon^+, x^+] = 0 \qquad \partial \frac{d}{d\alpha}[\varepsilon^+, x^+] = 0$$

$$\partial[\varepsilon^+, x^+] = q^A[\varepsilon^-, x^-] \qquad \frac{d}{d\alpha}\partial[\varepsilon^+, x^+] = \alpha(A) q^A [\varepsilon^-, x^-]$$

On $QH^*_{S^1}(M,\mathrm{Id}) = \Lambda \otimes \mathbb{Z}[\mathbf{u}] \otimes H^*(M)$, the **Dubrovin connection** is

$$\nabla_\alpha(q^A x) = \mathbf{u}\frac{d}{d\alpha}(q^A)x + \alpha * q^A x. \quad (5)$$

Theorem (Connection, TL-J)

On $FH^*_{S^1}(M,\lambda;H)$, for $\alpha \in H^2(M)$ the map

$$\nabla_\alpha = \mathbf{u}\frac{d}{d\alpha} + (\alpha * \cdot) - w_\alpha \quad (6)$$

is a chain map on the equivariant Floer cochain complex.

Seidel proved special case $\alpha = [\omega]$ in 2016.

Theorem (Flatness, TL-J)

On $FH^*_{S^1}(M, \lambda; H)$, for any $\alpha, \beta \in H^2(M)$, we have
$$\nabla_\alpha \circ \nabla_\beta = \nabla_\beta \circ \nabla_\alpha.$$

The Dubrovin connection is $\nabla_\alpha = \mathbf{u}\frac{d}{d\alpha} + \alpha *$.
A calculation shows the intertwining relation
$$Q\mathcal{S}_{S^1}(\sigma)(x * \alpha^+) - Q\mathcal{S}_{S^1}(\sigma)(x) * \alpha^- = \mathbf{u}\ WQ\mathcal{S}_{S^1}(\sigma, \alpha)(x)$$
is equivalent to
$$Q\mathcal{S}_{S^1}(\sigma)(\nabla_\alpha(x)) - \nabla_\alpha(Q\mathcal{S}_{S^1}(\sigma)(x)) = 0.$$

Theorem (Connection and Floer Seidel map, TL-J)

On $FH^*_{S^1}(M, \lambda; H)$, for any $\alpha \in H^2(M)$ and any linear σ, we have
$$\nabla_\alpha \circ F\mathcal{S}_{S^1}(\sigma) = F\mathcal{S}_{S^1}(\sigma) \circ \nabla_\alpha.$$

Now let T be a torus acting on M.
S^1-actions correspond to **cocharacters** of T, which are group maps $\sigma : S^1 \to T$.
Let $\widehat{T} = S^1_0 \times T$.
Constructions of $QH^*_{\widehat{T}}(M)$, $FH^*_{\widehat{T}}(M, \lambda)$, $SH^*_{\widehat{T}}(M)$.
We get ∇_α, $Q\mathcal{S}_{\widehat{T}}(\sigma)$, $F\mathcal{S}_{\widehat{T}}(\sigma)$ too.
But now we can undo the change of $\widehat{T}$-action, to get endomorphisms

$$\begin{aligned} \mathbb{S}_\sigma &: QH^*_{\widehat{T}}(M) \to QH^{*+|\sigma|}_{\widehat{T}}(M) \\ \mathbb{S}_\sigma &: SH^*_{\widehat{T}}(M) \to SH^{*+|\sigma|}_{\widehat{T}}(M) \end{aligned} \tag{7}$$

called **shift operators**.

The torus T^2 acts on $\mathbb{P}^2$ (on middle and last coordinate).
We have $H^*(BT) = \mathbb{Z}[t_1, t_2]$ and $\Lambda = \mathbb{Z}[q]$ with $|q| = 6$.

$$QH^*_{\widehat{T}}(\mathbb{P}^2) = \frac{\Lambda \otimes \mathbb{Z}[x_0, x_1, x_2, \mathbf{u}]}{x_0 x_1 x_2 - q} \tag{8}$$

We calculate $\nabla_x = \mathbf{u}(q\frac{d}{dq}) + x_0$.
Let σ correspond to rotation of middle coordinate. We have:

$$\begin{aligned} \mathbb{S}_\sigma(1) &= x_1 & \mathbb{S}_\sigma(t_1 y) &= (t_1 + \mathbf{u})\mathbb{S}_\sigma(y) \\ \mathbb{S}_\sigma(x_0) &= x_1 x_0 & \mathbb{S}_\sigma(t_2 y) &= t_2 \mathbb{S}_\sigma(y) \\ \mathbb{S}_\sigma(x_1) &= x_1(x_1 - \mathbf{u}) & & \\ \mathbb{S}_\sigma(x_2) &= x_1 x_2 & & \end{aligned} \tag{9}$$

We had the **Seidel representation**

$$\begin{aligned} \left\{\text{Hamiltonian } S^1\text{-actions}\right\} &\to QH^*(M)^\times \\ \sigma &\mapsto Q\mathcal{S}(\sigma)(1). \end{aligned} \tag{10}$$

We also have $\mathbb{S}_\sigma \mathbb{S}_{\sigma'} = \mathbb{S}_{\sigma+\sigma'}$.
This yields

$$\mathbb{S} : \mathrm{Cochar}^+(T) \to \mathrm{End}_{\Lambda[\mathbf{u}]}(QH^*_{\widehat{T}}(M))$$

$$\mathbb{S} : \mathrm{Cochar}(T) \to \mathrm{Aut}_{\Lambda[\mathbf{u}]}(SH^*_{\widehat{T}}(M)).$$

We have expanded the algebraic structures on $SH^*_{\widehat{T}}(M)$ with

- a flat connection ∇_α,
- shift operators $\mathbb{S}_\sigma$.

They are compatible, computable in examples and capture geometric information.

Thanks for your attention.

Definition (Borel construction of S^1-equivariant cohomology)

X is topological space, ρ is S^1-action on X.
Take S^∞: it's a contractible space with a free S^1-action.

Borel quotient is $S^\infty \times X/\sim$, where $\sim$ is $(\theta \cdot w, x) \sim (w, \rho_\theta(x))$.
It's denoted $S^\infty \times_{S^1} X$.

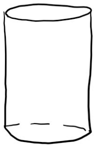
S^1-**equivariant cohomology** is $H^*_{S^1}(X, \rho) = H^*(S^\infty \times_{S^1} X)$.

The projection map $S^\infty \times_{S^1} X \to S^\infty / S^1 = \mathbb{CP}^\infty$ induces a map $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[\mathbf{u}] \to H^*_{S^1}(X, \rho)$.

The **Floer cohomology** $FH^*(M)$ is inspired by Morse cohomology on the loop space $\mathcal{L}M = \{x : S^1 \to M\}$.

Take a function $H : S^1 \times M \to \mathbb{R}$, called a **Hamiltonian function**.
Define the **Hamiltonian vector field** by $\omega(\cdot, X_t) = \mathrm{d}H_t$.
The **Hamiltonian orbits** are the curves $x : S^1 \to M$ that follow X_t.
The **Floer cochain complex** is freely generated over Λ by the Hamiltonian orbits.

A **Floer cylinder** is a cylinder $u : \mathbb{R} \times S^1 \to M$ which satisfies a **Floer equation**.
The **Floer differential** counts Floer cylinders between the Hamiltonian orbits.



Definition (Convex symplectic manifold)

A **convex symplectic manifold** is the union of a compact symplectic manifold and the symplectic manifold $([1, \infty) \times \Sigma, \mathrm{d}(R\alpha))$, where (Σ, α) is a closed contact manifold.

Example $\mathcal{O}_{\mathbb{P}^1}(-1)$, where $\Sigma = S^3$ is the sphere bundle.

Floer cohomology depends on the **slope** λ, where $H = \lambda R +$ constant at infinity.

Symplectic cohomology $SH^*(M)$ is the limit of $FH^*(M, \lambda)$ as $\lambda \to \infty$.